Nonlinear Suboptimal Tracking Controller Design Using State-Dependent Riccati Equation Technique

Yazdan Batmani, Mohammadreza Davoodi, and Nader Meskin

***Abstract*— In this brief, a new technique for solving a suboptimal tracking problem for a class of nonlinear dynamical systems is presented. Toward this end, an optimal tracking problem using a discounted cost function is defined and a control law with a feedback-feedforward structure is designed. A state-dependent Riccati equation (SDRE) framework is used in order to find the gains of both the feedback and the feedforward parts, simultaneously. Due to the significant properties of the SDRE technique, the proposed method can handle the presence of input saturation and state constraint. It is also shown that the tracking error converges asymptotically to zero under mild conditions on the discount factor of the corresponding cost function and the desired trajectory. Two simulation and experimental case studies are also provided to illustrate and demonstrate the effectiveness of our proposed design methodology.**

***Index Terms*— Input saturation, linear quadratic tracking, optimal control, state-dependent Riccati equation (SDRE), time-varying desired trajectory.**

I. Introduction

OPTIMAL control deals with the problem of finding a control law in order to achieve the best possible behavior with respect to a predefined criterion. The optimal quadratic regulation problem for linear systems was solved in the 1960s [1] and the obtained results were also extended to the optimal tracking problem for linear systems [1], [2]. Nevertheless, in many practical engineering problems, the system to be controlled is nonlinear. Due to the complexity of the arising Hamilton–Jacobi–Bellman (HJB) equation, which is too difficult or even impossible to be solved, various methods were developed to find approximate solutions of the nonlinear optimal regulation problem (see [3]–[5]). Although some methods were proposed to solve the optimal tracking problem for nonlinear systems [6], [7], it can be said that much less attention has been paid to this problem.

The state-dependent Riccati equation (SDRE) technique was originally proposed by Pearson in 1962 to approximately solve the optimal regulation problem for nonlinear systems [8]. Representing a nonlinear system dynamics as a state-dependent linear system, called the pseudo-linearization or extended linearization [9], is the main idea of the SDRE technique. Since then several methods have been developed based on the pseudo-linearization framework to solve different problems such as robust H_∞ filter design [10], suboptimal sliding mode control design for delayed systems [11], observer design for nonlinear delayed systems [12], and so on. These methods were effectively applied in a wide variety of applications, such as drug administration in cancer treatment [13] and dive plane control of autonomous underwater vehicles (AUVs) [14]. Two complete surveys of the SDRE techniques and the related theories can be found in [8] and [9].

For the set-point tracking problem, the SDRE technique is developed based on the integral action method [8]. However, to the best of our knowledge, the optimal tracking control problem for nonlinear systems, which is practically very important, has not been solved using the SDRE technique. The main reason for this shortage is that the quadratic cost function used in the SDRE technique is only valid for the desired trajectories generated by an asymptotically stable system. However, many of desired trajectories, such as steps and sinusoidal signals, are not generated by such systems. This problem and interesting properties of the SDRE method, such as simplicity and flexibility of the SDRE design procedure, ability to consider input saturation, and maintaining the nonlinear characteristics of the system, motivate us to develop an SDRE-based control design method for the nonlinear tracking problem.

Toward this end, a discounted cost function is used to tackle the above-mentioned problem and define an optimal tracking problem for more general desired trajectories. Then, the optimal nonlinear tracking problem is converted into an optimal nonlinear regulation problem and the SDRE technique is used to find a suboptimal solution of the obtained optimal regulation problem or equivalently a solution of the original optimal tracking problem. The proposed method inherits almost all of the interesting properties of the SDRE technique such as ability to consider input saturation, robustness with respect to parametric uncertainties and unmodelled dynamics, and so on. The preliminary result of this brief is presented in [15]. In this brief, the stability of the proposed tracking controller is investigated and a theorem is also presented to find proper values of the discount factor. The results of applying the proposed method to two simulation and experimental case studies are also presented to illustrate the effectiveness and capabilities of the proposed design methodology.

The remainder of this brief is organized as follows. In Section II, we first define an optimal tracking problem for a broad class of nonlinear dynamical systems and then using the pseudo-linearization technique, a method is proposed to find a suboptimal solution of the tracking problem. The asymptotic stability of the closed-loop system is also investigated in this section. In Section III, results of applying the proposed method

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to two practical case studies (dive plane control of an AUV and level control of a three-tank system) are presented. Finally, Section IV concludes this brief.

II. Controller Design Methodology

A. System Description and Problem Statement

Consider the following nonlinear dynamical system:

$$\begin{aligned}\dot{x}(t) &= f(x(t)) + b(x(t))u(t), \quad x(0) = x_0\\ y(t) &= h(x(t))\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, and $y(t) \in \mathbb{R}^p$ is the system output. $f : \mathbb{R}^n \to \mathbb{R}^n$, $b : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $h : \mathbb{R}^n \to \mathbb{R}^p$ are assumed to be smooth functions and $f(0) = h(0) = 0$ and $b(x) \neq 0$ for all $x \in \mathbb{R}^n$.

As mentioned, the optimal tracking problem with traditional quadratic cost function is only valid for the cases where the desired trajectories are generated by an asymptotically stable system. However, there are many practically important trajectories, which are not generated by such a system. In this brief, to tackle this problem, a discounted cost function is considered and a technique to solve the following optimal tracking problem is proposed.

Discounted Infinite-Time Horizon Nonlinear Optimal Tracking (DITHNOT) Problem:

Find the control input $u(t)$, $t \geq 0$, such that the system output $y(t)$, $t \geq 0$ tracks the desired trajectory $y_d(t)$, $t \geq 0$, and the following discounted cost function is minimized:

$$\begin{aligned}J(x_0, u(t), y_d(t)) = \int_0^\infty e^{-2\gamma t}((y(t) - y_d(t))^T Q_1(y(t)\\ - y_d(t)) + u^T(t)Ru(t))dt\end{aligned} \tag{2}$$

where $\gamma > 0$ is the discount factor. It is assumed that Q_1 and R are, respectively, positive-semidefinite and positive-definite symmetric matrices with appropriate dimensions. Assume further that the desired trajectory has the general nonlinear dynamics

$$\begin{aligned}\dot{x}_d(t) &= f_d(x_d(t)), \quad x_d(0) = x_{d0}\\ y_d(t) &= h_d(x_d(t))\end{aligned} \tag{3}$$

where $x_d(t) \in \mathbb{R}^{n_d}$ and $y_d(t) \in \mathbb{R}^p$ are the state and output of the desired trajectory system (3) and functions $f_d : \mathbb{R}^{n_d} \to \mathbb{R}^{n_d}$ and $h_d : \mathbb{R}^{n_d} \to \mathbb{R}^p$ are assumed to be smooth and $f_d(0) = h_d(0) = 0$. Note that many useful desired trajectories, such as steps, sinusoidal signals, and damped sinusoids, can be generated by (3).

Applying Bellman's principle of optimality to the above DITHNOT problem leads to an HJB equation, which is too difficult or impossible to be analytically solved. Therefore, finding approximate solutions of the problem is considered as an alternative way in order to avoid encountering the complicated HJB equation. In Section II-B, based on the pseudo-linearization idea, a technique to find a suboptimal solution of the DITHNOT problem is proposed.

B. Proposed Method

Since the nonlinear functions f, h, f_d, and h_d are assumed to be smooth and $f(0) = h(0) = f_d(0) = h_d(0) = 0$, they can be rewritten in their pseudo-linearized forms (also called state-dependent coefficient (SDC)) as follows [8]:

$$\begin{aligned}f(x(t)) &= F(x(t))x(t), \quad f_d(x_d(t)) = F_d(x_d(t))x_d(t)\\ h(x(t)) &= H(x(t))x(t), \quad h_d(x_d(t)) = H_d(x_d(t))x_d(t)\end{aligned} \tag{4}$$

where $F : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $H : \mathbb{R}^n \to \mathbb{R}^{p\times n}$, $F_d : \mathbb{R}^{n_d} \to \mathbb{R}^{n_d\times n_d}$, and $H_d : \mathbb{R}^{n_d} \to \mathbb{R}^{p\times n_d}$ are four matrix-valued functions. It should be mentioned that there are infinite ways to pseudo-linearize non-scalar systems. This property of the pseudo-linearization technique provides additional degrees of freedom, which can enhance the design procedure of SDRE-based methods [13].

Defining $X(t) \triangleq e^{-\gamma t}\left[x^T(t)\ x_d^T(t)\right]^T \in \mathbb{R}^{n+n_d}$ and $U(t) \triangleq e^{-\gamma t}u(t)$ and substituting them in the cost function (2) in the DITHNOT problem leads to

$$J(X_0, U(t)) = \int_0^\infty (X^T(t)Q(e^{\gamma t}X(t))X(t) + U^T(t)RU(t))dt \tag{5}$$

where

$$\begin{aligned}Q(e^{\gamma t}X(t)) = [H(x(t)) \quad &- H_d(x_d(t))]^T\\ Q_1[H(x(t)) \quad &- H_d(x_d(t))].\end{aligned}$$

The nonlinear dynamics of $X(t)$ is given as

$$\dot{X}(t) = -\gamma X(t) + e^{-\gamma t}\left[\dot{x}^T(t) \quad \dot{x}_d^T(t)\right]^T.$$

Now, by substituting $\dot{x}(t)$ and $\dot{x}_d(t)$ from (1) and (3), respectively, and using (4), we have the following augmented pseudo-linearized dynamics:

$$\begin{aligned}\dot{X}(t) &= \left(-\gamma I + \begin{bmatrix} F(x(t)) & 0\\ 0 & F_d(x_d(t))\end{bmatrix}\right)X(t) + \begin{bmatrix} b(x(t))\\ 0\end{bmatrix}\\ U(t) &= A(e^{\gamma t}X(t))X(t) + B(e^{\gamma t}X(t))U(t)\end{aligned} \tag{6}$$

where I and 0 denote the identity and zero matrices with appropriate dimensions, respectively. Therefore, an infinite-time horizon nonlinear optimal regulation (ITHNOR) problem, described by (5) and (6), should be solved instead of the DITHNOT problem. The optimal solution of the ITHNOR problem is

$$U(t) = -R^{-1}B(e^{\gamma t}X(t))\frac{\partial V(t, X(t))}{\partial X(t)}$$

where $V(t, X(t))$ is the solution of the following HJB equation, which arises from Bellman's principle of optimality [9]:

$$\begin{aligned}-\frac{\partial V}{\partial t} = \inf_U\Bigg(\left(\frac{\partial V}{\partial X}\right)^T \dot{X}(t) + X^T(t)Q(e^{\gamma t}X(t))X(t)\\ + U^T(t)RU(t)\Bigg).\end{aligned} \tag{7}$$

However, solving the above HJB equation is not generally easier than the HJB equation arising from the original DITHNOT problem. Nevertheless, there are some well-known approximation methods to solve ITHNOR problems [3]–[5], and among them, the SDRE is one of the most popular methods, which yield to suboptimal performance [9]. In the following, the SDRE technique is used to find a suboptimal control law for the ITHNOR problem, or equivalently the DITHNOT problem. Toward this end, some necessary definitions, which are needed in the rest of this brief, are presented.

Definition 1: The SDC representation (6) is pointwise stabilizable in the bounded open set $\Omega \in \mathbb{R}^{n+n_d}$ if the pair $(A(e^{\gamma t}X(t)), B(e^{\gamma t}X(t)))$ is stabilizable in the linear sense for all $X(t) \in \Omega$ and $t \geq 0$.

Definition 2: The SDC representation (6) is pointwise detectable in the bounded open set $\Omega \in \mathbb{R}^{n+n_d}$ if the pair $(A(e^{\gamma t}X(t)), Q^{1/2}(e^{\gamma t}X(t)))$ is detectable in the linear sense for all $X(t) \in \Omega$ and $t \geq 0$.

In order to find a suboptimal solution for the above ITHNOR problem using the SDRE technique, two steps must be taken [16]. At the first step, the following state-dependent algebraic Riccati equation:

$$\begin{aligned} &A^T(e^{\gamma t}X(t))P(e^{\gamma t}X(t)) + P(e^{\gamma t}X(t))A(e^{\gamma t}X(t)) \\ &\quad - P(e^{\gamma t}X(t))B(e^{\gamma t}X(t))R^{-1}B^T(e^{\gamma t}X(t))P(e^{\gamma t}X(t)) \\ &\quad + Q(e^{\gamma t}X(t)) = 0 \end{aligned} \tag{8}$$

should be solved to find the matrix $P(e^{\gamma t}X(t))$. The SDRE (8) has a unique symmetric positive-definite solution $P(e^{\gamma t}X(t))$ if the triple $(A(e^{\gamma t}X(t)), B(e^{\gamma t}X(t)), Q^{1/2}(e^{\gamma t}X(t)))$ is point-wise stabilizable and detectable [16]. While this equation can be solved analytically for simple problems, there are some numerical methods to find its solution for complex systems [8]. The second step of the SDRE design procedure is to compute the control law $U(t)$ as

$$U(t) = -R^{-1}B^T(e^{\gamma t}X(t))P(e^{\gamma t}X(t))X(t). \tag{9}$$

It can be seen that the above technique uses the solution of the SDRE (8) instead of solving the HJB equation (7). Although it has been shown [9] that there is always an SDC representation, which yields to the optimal solution, finding such an SDC form is not straightforward. However, using any SDC representation leads to having a suboptimal control law. The following theorem shows that under which conditions the SDRE technique leads to a locally stable closed-loop system.

Theorem 1: Assume that the triple $(A(e^{\gamma t}X(t)), B(e^{\gamma t}X(t)), Q^{1/2}(e^{\gamma t}X(t)))$ is pointwise stabilizable and detectable in the bounded open set $\Omega \in \mathbb{R}^{n+n_d}$ where $0 \in \Omega$. The control law (9) guarantees the local asymptotic stability of the origin of the system (6), where $P(e^{\gamma t}X(t))$ is the solution of the SDRE (8).

Proof: Due to the point-wise stabilizability and detectability of the SDC representation (6), from Riccati equation theory, it can be concluded that the SDRE (8) has a unique symmetric, positive-definite solution $P(e^{\gamma t}X(t))$ [11]. Using Taylor series expansion, $A(e^{\gamma t}X(t))$, $B(e^{\gamma t}X(t))$, and $P(e^{\gamma t}X(t))$ can be written as $A(e^{\gamma t}X(t)) = A_0 + \Delta A(e^{\gamma t}X(t))$, $B(e^{\gamma t}X(t)) = B_0 + \Delta B(e^{\gamma t}X(t))$, and $P(e^{\gamma t}X(t)) = P_0 + \Delta P(e^{\gamma t}X(t))$, where $A_0 = A(0)$, $B_0 = B(0)$, and $P_0 = P(0)$. $\Delta A(e^{\gamma t}X(t))$, $\Delta B(e^{\gamma t}X(t))$, and $\Delta P(e^{\gamma t}X(t))$ are the other terms of Taylor series expansions for $A(e^{\gamma t}X(t))$, $B(e^{\gamma t}X(t))$, and $P(e^{\gamma t}X(t))$, respectively. Applying the control law (9) leads to the following closed-loop system:

$$\begin{aligned} \dot{X}(t) &= A(e^{\gamma t}X(t))X(t) - B(e^{\gamma t}X(t))R^{-1} \\ &\qquad B^T(e^{\gamma t}X(t))P(e^{\gamma t}X(t))X(t) \triangleq A^{\text{cl}}(e^{\gamma t}X(t))X(t). \end{aligned}$$

The closed-loop system dynamics $A^{\text{cl}}(e^{\gamma t}X(t))$ can be rewritten as $A^{\text{cl}}(e^{\gamma t}X(t)) = A_0^{\text{cl}} + \Delta A^{\text{cl}}(e^{\gamma t}X(t))$, where

$$A_0^{\text{cl}} = A_0 - B_0R^{-1}B_0^TP_0. \tag{10}$$

Since $\Delta A(e^{\gamma t}X(t))$, $\Delta B(e^{\gamma t}X(t))$, and $\Delta P(e^{\gamma t}X(t))$ tend to zero for the small values of $X(t)$, one can see that $\Delta A^{\text{cl}}(e^{\gamma t}X(t))$ tends to zero. Now, consider the Lyapunov function $V(X(t)) = X^T(t)P_0X(t)$. The derivative of $V(X(t))$ along the trajectory $\dot{X}(t) = A^{\text{cl}}(e^{\gamma t}X(t))X(t)$ is given as

$$\begin{aligned} \dot{V}(X(t)) &= X^T(t)((A^{\text{cl}})^TP_0 + P_0A^{\text{cl}})X(t) \\ &= X^T(t)\big(\big(A_0^{\text{cl}}\big)^TP_0 + P_0A_0^{\text{cl}} + \sigma(X(t))\big)X(t) \end{aligned}$$

where $\sigma(X(t)) = (\Delta A^{\text{cl}}(e^{\gamma t}X(t)))^TP_0 + P_0\Delta A^{\text{cl}}(e^{\gamma t}X(t))$.

Substituting A_0^{cl} from (10) in the above equality leads to

$$\dot{V}(X(t)) = -X^T(t)\tilde{Q}(X(t))X(t) + X^T(t)\sigma(X(t))X(t) \tag{11}$$

where $\tilde{Q}(X(t)) \triangleq Q(X(t)) + P_0B_0R^{-1}B_0^TP_0$. Since $\Delta A^{\text{cl}}(e^{\gamma t}X(t))$ tends to zero for small values of $X(t)$, $\sigma(X(t))$ tends to zero, and therefore, for any $\epsilon > 0$, there exists $\delta > 0$, such that $\|\sigma(X(t))\| < \epsilon$ for all $X(t) \in B_\delta \triangleq \{X(t) \mid \|X(t)\| < \delta\}$. From (11), it can be concluded that

$$\dot{V}(X(t)) < -X^T(t)\tilde{Q}(X(t))X(t) + \epsilon\|X(t)\|^2 \quad \forall X(t) \in B_\delta.$$

On the other hand, if $\lambda_{\min}(.)$ denotes the minimum eigenvalue of a matrix, the following inequality holds:

$$\dot{V}(X(t)) < -(\lambda_{\min}(\tilde{Q}(X(t))) - \epsilon)\|X(t)\|^2 \quad \forall X(t) \in B_\delta.$$

Since $\tilde{Q}(X(t))$ is symmetric and positive definite for $X(t) \in B_\delta$, $\lambda_{\min}(\tilde{Q}(X(t)))$ is positive [17]. Therefore, $\dot{V}(X(t))$ is negative by selecting

$$\epsilon < \inf_{X(t)\in\Omega}(\lambda_{\min}(\tilde{Q}(X(t))))$$

and this completes the proof. □

From Theorem 1, using the proposed method, the augmented state variable $X(t) = e^{-\gamma t}[x^T(t)\ x_d{}^T(t)]^T$ asymptotically tends to zero and the cost function (2) is minimized in a suboptimal way. Therefore, it can be concluded that the tracking error $e(t) = y(t) - y_d(t)$ tends to zero for $\gamma \to 0$.

Using the obtained control law $U(t)$, we can find the following control law for the original DITHNOT problem:

$$\begin{aligned} &u(x(t), x_d(t)) \\ &\quad = -R^{-1}B^T(x(t))P(x(t), x_d(t))\big[x(t)^T \quad x_d{}^T(t)\big]^T \end{aligned} \tag{12}$$

which can be rewritten as $u(x(t), x_d(t)) = -K_f(x(t), x_d(t))\, x(t) - K_{ff}(x(t), x_d(t))x_d(t)$, where $K_f(x(t), x_d(t))$ and $K_{ff}(x(t), x_d(t))$ are, respectively, the state-dependent feedback and feedforward gains, which both are calculated from solving the SDRE (8). The following theorem shows

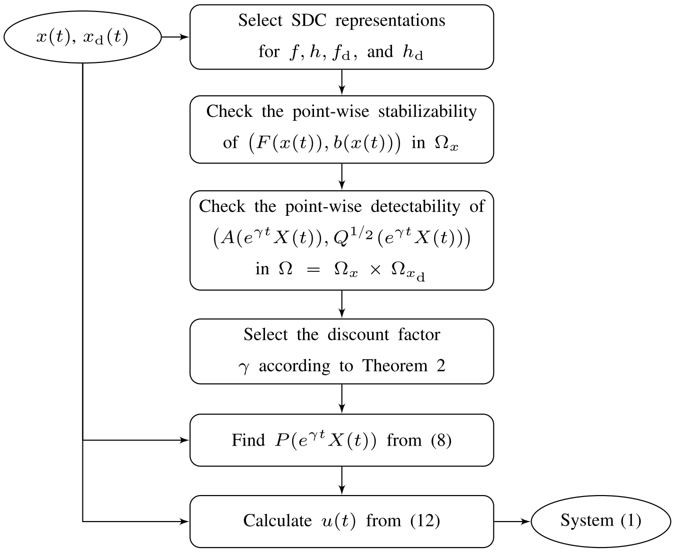

Fig. 1. Flowchart of the proposed tracking controller.

that it is possible to find a lower bound for γ in such a way that the SDRE (8) can be solved.

Theorem 2: The SDC representation (6) is pointwise stabilizable in $\Omega = \Omega_x \times \Omega_{x_d}$ if the pair $(F(x), b(x))$ is pointwise stabilizable in $\Omega_x \subset \mathbb{R}^n$ and the discount factor γ satisfies

$$\gamma > \sup_{x_d \in \Omega_{x_d}} (\max(\mathrm{Re}(\lambda(F_d(x_d)))))$$

where $\lambda(F_d(x_d))$ is the eigenvalue of $F_d(x_d)$ for $x_d \in \Omega_{x_d}$.

Proof: The state-dependent controllability matrix of the pair $\left(A(e^{\gamma t}X(t)), B(e^{\gamma t}X(t))\right)$ is as follows:

$$\Phi_c = \begin{bmatrix} b(x) & (F(x)-\gamma I)b(x) & \dots & (F(x)-\gamma I)^{n+n_d-1}b(x) \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

It can be concluded that if the pair $(F(x), b(x))$ is pointwise stabilizable in Ω_x, then the above controllability matrix Φ_c has a rank of n and the state variables $X_i, i = 1, \dots, n$ are controllable. On the other hand, the remaining state variables are uncontrollable. Note that this result is trivial, since these states are actually related to the desired trajectory. Nevertheless, if all the real parts of the eigenvalues of the state-dependent matrix $F_d(x_d) - \gamma I$ are negative for all $x_d \in \Omega_{x_d}$, then these states are pointwise stabilizable in Ω_{x_d}. This condition is guaranteed if the real parts of all the eigenvalues of $F_d(x_d)$ are smaller than γ. This completes the proof. □

From Theorems 1 and 2, it can be concluded that the discount factor γ is a critical parameter of the proposed method. Although the tracking error $e(t)$ only guarantees to be zero for small values of γ, in some cases due to Theorem 2, we have to select larger values for the discount factor γ, which causes error in the tracking. However, the observed error can be decreased by selecting larger values for the elements of the weighing matrix Q_1. The main steps involved in the computation of the proposed SDRE tracking controller are illustrated in Fig. 1.

Remark 1: The proposed SDRE tracking controller can be used in a class of nonlinear delayed systems based on the extensions of the SDRE regulator in [18].

III. Case Studies

In this section, the proposed SDRE tracking controller is applied to two practical examples. The first one demonstrates how the proposed tracking controller can solve the problem of dive plane control of an AUV in a complex mission. The second example concerns the problem of level control of a laboratory three-tank system.

A. Dive Plane Control of AUV

AUVs have become an increasingly important tool in a number of applications over the recent years such as deep sea inspections, neutralize undersea mines, and so on. Design of controllers for AUVs is an extremely difficult task mostly due to the inherent nonlinearity of the underwater vehicle dynamics. On the other hand, it is so important to design a controller so as to make the AUV tracks a desired time-varying trajectory in complex missions in order to avoid hitting physical obstacles. The considered AUV in this brief is a REMUS AUV, which has been described in detail by [19]. The objective of this example lies in the design of a robust suboptimal tracking control system for the control of AUVs in the dive plane using the proposed method presented in Section II. For this purpose, the following SDC representation of the AUV model is used [14]:

$$\begin{aligned} \dot{\mathcal{X}}(t) &= \begin{bmatrix} \mathcal{A}_1(\mathcal{X}(t)) & \mathcal{A}_2(\mathcal{X}(t)) \\ \mathcal{A}_3(\mathcal{X}(t)) & \mathcal{A}_4(\mathcal{X}(t)) \end{bmatrix} \mathcal{X}(t) + \begin{bmatrix} b_1 \\ 0_{2\times 1} \end{bmatrix} \delta_s(t) \\ &\quad + \begin{bmatrix} d_1 \\ 0_{2\times 1} \end{bmatrix} = \mathcal{A}(\mathcal{X}(t))\mathcal{X}(t) + b_2\delta_s(t) + d \qquad (13) \end{aligned}$$

where $\mathcal{X}(t) = [w(t)\ q(t)\ z(t)\ \theta(t)]^T$

$$\begin{aligned} \mathcal{A}_1(\mathcal{X}(t)) &= M^{-1} \begin{bmatrix} Z_{uw}u + Z_{w|w|}|w(t)| & \mathcal{A}_{1,12}(\mathcal{X}(t)) \\ M_{uw}u + M_{w|w|}|w(t)| & \mathcal{A}_{1,22}(\mathcal{X}(t)) \end{bmatrix} \\ \mathcal{A}_2(\mathcal{X}(t)) &= M^{-1} \begin{bmatrix} 0 & \mathcal{A}_{2,12}(\mathcal{X}(t)) \\ 0 & \mathcal{A}_{2,22}(\mathcal{X}(t)) \end{bmatrix} \\ \mathcal{A}_3(\mathcal{X}(t)) &= \begin{bmatrix} \cos(\theta(t)) & 0 \\ 0 & 1 \end{bmatrix}, \quad d_1 = \begin{bmatrix} W - B_0 \\ x_B B_0 - x_G W \end{bmatrix} \\ \mathcal{A}_4(\mathcal{X}(t)) &= \begin{bmatrix} 0 & -u\sin(\theta(t))\theta^{-1}(t) \\ 0 & 0 \end{bmatrix}, \quad b_1 = M^{-1} \begin{bmatrix} Z_{uu} \\ M_{uu} \end{bmatrix} u^2 \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}_{1,12}(\mathcal{X}(t)) &= Z_{uq} + Z_{q|q|}|q(t)| + mz_G q(t) + mu \\ \mathcal{A}_{1,22}(\mathcal{X}(t)) &= M_{uq} + Z_{q|q|}|q(t)| - m(\cos(\theta(t)) - 1)\theta^{-1}(t) \\ \mathcal{A}_{2,12}(\mathcal{X}(t)) &= (W - B_0)(\cos(\theta(t)) - 1)\theta^{-1}(t) \\ \mathcal{A}_{2,22}(\mathcal{X}(t)) &= (x_B B_0 - x_G W)(\cos(\theta(t)) - 1)\theta^{-1}(t) \\ &\quad - (z_G W - z_B)\sin(\theta(t))\theta^{-1}(t) \\ M &= \begin{bmatrix} m - Z_{\dot{w}} & -mx_G - Z_{\dot{q}} \\ -mx_G - M_{\dot{w}} & I_{yy} - M_{\dot{q}} \end{bmatrix}. \end{aligned}$$

In the above equations, w, q, z, and θ are the heave velocity, the pitch velocity, the depth, and the pitch angle, respectively, and δ_s denotes the fin angle, which is considered as the control input for the dive plane control of the AUV. The hydrodynamic parameters values of the AUV are reported in Table I [14]. Table II also represents the physical parameters values of the AUV [14], where (x_B, z_B) and (x_G, z_G) are the coordinates of the center of buoyancy and the coordinates of the center

TABLE I
HYDRODYNAMIC PARAMETERS OF THE REMUS

Parameter	Value	Parameter	Value
$M_{\dot{q}}$	-4.88 kg m^2/rad	$M_{\dot{w}}$	-1.93 kg m
$M_{q\|q\|}$	-188 kg m^2/rad	$M_{w\|w\|}$	3.18 kg
M_{uq}	-2 kg m/rad	M_{uw}	24 kg
$Z_{\dot{q}}$	-1.93 kg m/rad	$Z_{\dot{w}}$	-35.5 kg
$Z_{q\|q\|}$	-0.632 kg m/rad^2	$Z_{w\|w\|}$	-131 kg/m
Z_{uq}	-5.22 kg/rad	Z_{uw}	-28.6 kg/m
M_{uu}	-6.15 kg/rad	Z_{uu}	-6.15 kg/(m rad)

TABLE II
VEHICLE PHYSICAL PARAMETERS OF THE REMUS

Parameter	Value	Parameter	Value
x_G	0	z_G	0.0196 m
x_B	0	z_B	0
W	299 N	B_0	306 N
m	30.48 kg	I_{yy}	3.45 kg m^2

of gravity of the AUV with respect to the center of buoyancy, respectively. W denotes the AUV's weight, B_0 is the vehicle buoyancy, m is the mass of the AUV, and I_{yy} is the moment of inertia of the AUV about the pitch axis. It should be mentioned that in the following simulations, the forward velocity u is assumed to be held as $u = 2$ m/s.

Since the amplitude of the control input, *i.e.*, the fin angle, cannot be larger than a certain value, it is so vital to consider the presence of the input saturation in the design procedure. Unlike the well-known nonlinear controller design techniques, such as sliding mode control and backstepping, the SDRE method can *easily* handle this problem [8]. Let us define an auxiliary input $\tilde{\delta}_s(t)$ and consider the following dynamics for the fin angle:

$$\dot{\delta}_s(t) = \tilde{\delta}_s(t). \tag{14}$$

Augmenting (13) and (14) yields to the following SDC representation:

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} \mathcal{A}(\mathcal{X}(t)) & b_2\delta_s^{-1}(t)\mathrm{sat}(\delta_s(t), \delta_{\mathrm{sm}}) \\ 0_{1\times 4} & 0 \end{bmatrix} x(t) \\ &\quad + \begin{bmatrix} 0_{4\times 1} \\ 1 \end{bmatrix} \tilde{\delta}_s(t) + \begin{bmatrix} d \\ 0 \end{bmatrix} \\ &= F(x(t))x(t) + b\tilde{\delta}_s(t) + D \end{aligned} \tag{15}$$

where $x(t) = [\mathcal{X}^T(t)\ \ \delta_s(t)]^T$, δ_{sm} is the maximum admissible value of the fin angle, and sat is the saturation function defined as follows:

$$\mathrm{sat}(\delta_s(t), \delta_{\mathrm{sm}}) = \begin{cases} \delta_{\mathrm{sm}}, & \delta_s(t) > \delta_{\mathrm{sm}} \\ \delta_s(t), & |\delta_s(t)| \le \delta_{\mathrm{sm}} \\ -\delta_{\mathrm{sm}}, & \delta_s(t) < -\delta_{\mathrm{sm}}. \end{cases}$$

For the dive plane control problem of the AUV, the output $z(t)$ is as follows:

$$z(t) = [\,0 \quad 0 \quad 1 \quad 0 \quad 0\,]x(t) = Hx(t). \tag{16}$$

Based on the above SDC representation, the problem of tracking a constant trajectory is solved using the SDRE method [14]. Nevertheless, it is so important to design a controller, such that the AUV tracks a desired time-varying trajectory. In the following simulations, it is assumed that due to some physical obstacles, the AUV has to track a damped sinusoidal trajectory, which can be described by the following dynamics:

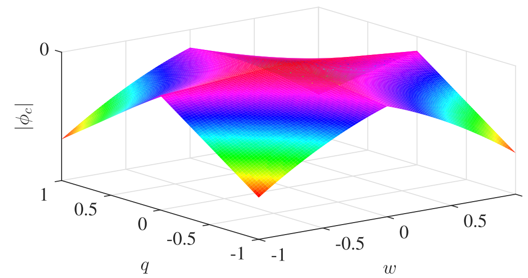

Fig. 2. Determinant of ϕ_c.

$$\begin{aligned}\dot{x}_d(t) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -0.5 & -0.5 \end{bmatrix} x_d(t) = F_d x_d(t), \\ z_d(t) &= [\,1 \quad 0 \quad 0\,]\, x_d(t) = H_d x_d(t), \\ x_d(0) &= [\,0 \quad 0 \quad 0.5\,]^T. \end{aligned} \tag{17}$$

Now, the above dive plane control of the AUV, represented by (15)–(17), can be considered as a tracking problem, which is in the form of (1) and (3). Therefore, it is possible to apply the proposed method, provided that the conditions of Theorem 1 are satisfied. To check the pointwise stabilizability condition, according to Theorem 2, the pair $(F(x), b)$ must be stabilizable in the domain of interest in the state space $\mathbb{R}^5$ and the discounted factor γ must be positive. For the pair $(F(x), b)$, the pointwise controllability matrix $\phi_c \in \mathbb{R}^{5\times 5}$ is as $\phi_c = [b\ \ F(x)b\ \ F^2(x)b\ \ F^3(x)b\ \ F^4(x)b]$.

A code in MATLAB is written to compute the determinant of ϕ_c in the domain $\Omega_x = \{x \in \mathbb{R}^5 : ||x|| \le 5\}$. This domain is considered based on the initial conditions of the AUV as well as the desired trajectory $z_d(t)$ in the following simulations. The obtained results show that in this domain, the determinant of ϕ_c is always negative, and therefore, the pair $(F(x), b)$ is controllable in Ω_x. Fig. 2 shows the determinant of the pointwise controllability ϕ_c when z, θ, and δ_s are assumed to be zero and $-1 \le w, q \le 1$.

The above results show that the pointwise stabilizability condition in Theorem 1 is satisfied provided that the discount factor γ is a positive constant. Paying attention to the stability of the desired trajectory (17), in the following simulations, this parameter is selected as $\gamma = 0.01$. The same analysis shows that the pointwise detectability condition in Theorem 1 is also satisfied and the augmented SDC representation (6) is pointwise detectable in the domain $\Omega \in \mathbb{R}^8$, which contains Ω_x. Therefore, the proposed tracking controller can be applied to the problem of dive plane control (see (15)–(17)). Fig. 3(a) shows the depth $z(t)$ when the amplitude of the fin angle δ_s is unconstrained as well as when it is assumed to be limited to $\delta_{\mathrm{sm}} = 40°$. In these simulations, the weighting matrices and the initial conditions are $Q_1 = 100$, $R = 0.01$, and $x(0) = [0\ 0\ 0\ 0\ 0]^T$, respectively. As it can be seen from

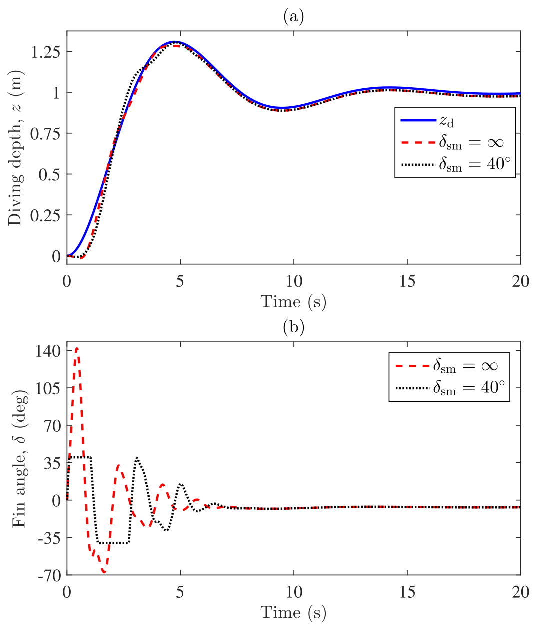

Fig. 3. Results of applying the proposed tracking controller to the AUV. (a) Depth of the AUV. (b) Control input.

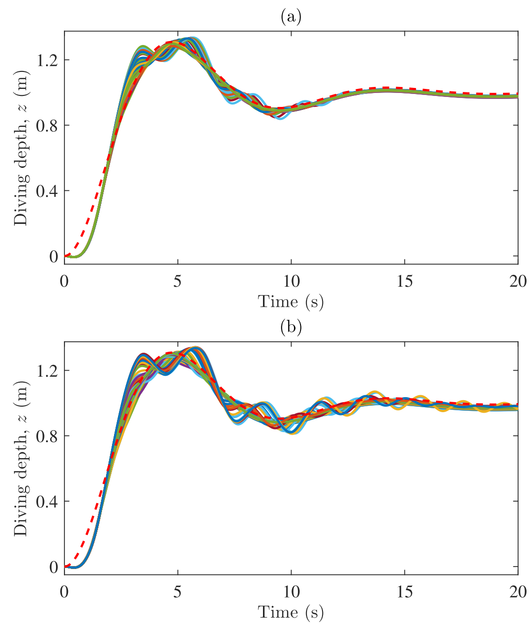

Fig. 4. Graphs of the depth of the AUV for (a) $\epsilon = 0.3$ and (b) $\epsilon = 0.5$. Dashed lines: desired trajectory.

Fig. 3(a), the obtained results are satisfactory and the AUV tracks the desired trajectory $z_d(t)$ even in the presence of input saturation. Fig. 3(b) shows the corresponding control inputs with and without input saturation.

To apply the proposed tracking controller to the AUV, the parameter values of the AUV model must be known. However, in practice, their actual values are different from their nominal values reported in Table I. Therefore, the designed controller must be robust against the uncertainty in the parameters to maintain its nominal performance. On the other hand, as it was mentioned in Section I, the SDRE controller has intrinsic robustness with respect to parametric uncertainties. Since the proposed tracking controller is the extension of the SDRE technique to the DITHNOT problem, it is expected to inherit this crucial property of the SDRE. To examine the robustness of the proposed tracking controller, it is assumed that the hydrodynamics parameters of the AUV are uncertain but bounded. Indeed, the controller is designed using the nominal parameters of the AUV (p^*) while the correct values of the parameters are randomly selected in the bound $[p^* - \epsilon p^*,\ p^* + \epsilon p^*]$. The simulation results for $\epsilon = 0.3$ and $\epsilon = 0.5$ are shown in Fig. 4. In each case, simulations are run for 250 sets of the parameters. For $\epsilon = 0.3$, the average and the standard deviation of the root mean square error are 0.0021 and 0.0045, respectively. These values are, respectively, increased to 0.0030 and 0.0049 for $\epsilon = 0.5$. It can be concluded that the proposed tracking controller is so robust against parametric uncertainties and the depth of the AUV successfully tracks the desired time-varying trajectory.

B. Level Control of a Three-Tank System

In this example, to illustrate the performance of the proposed controller, a setup of a three-tank system with mathematical model $A_t\dot{h}_1(t) = q_1(t) - q_{13}(t)$, $A_t\dot{h}_3(t) = q_{13}(t) - q_{32}(t)$, and $A_t\dot{h}_2(t) = q_2(t) + q_{32}(t) - q_{20}(t)$ is considered, where h_i denotes the level of tank i in m, $(i = 1, 2, 3)$. q_1 and q_2 are the supplying flow rates in $\mathrm{m}^3/\mathrm{sec}$, q_{ij} shows the water flow from tank i to tank j in $\mathrm{m}^3/\mathrm{sec}$, $(i, j \in \{1, 2, 3\})$, and A_t denotes the section of the cylinder in m^2 [20]. The three-tank system has four state regions in which the corresponding model is differentiable [20]. In this example, we consider the region $h_1(t) > h_3(t) > h_2(t)$. Using the generalized Torricelli rule, equations $q_{13}(t) = a_1 S(2g(h_1(t) - h_3(t)))^{1/2}$, $q_{32}(t) = a_3 S(2g(h_3(t) - h_2(t)))^{1/2}$, and $q_{20}(t) = a_2 S(2gh_2(t))^{1/2}$ are obtained for the flow rates, where g is the earth acceleration in $\mathrm{m}/\mathrm{sec}^2$, S and a_i $(i \in \{1, 2, 3\})$, respectively, denote the section of the connection pipes in m^2 and the outflow coefficients [20].

The control problem is to find the control law $u(t) = [q_1(t)\ \ q_2(t)]^T$ in such a way that the level of the first and the second tanks is set to some predefined values. To implement the proposed SDRE tracking controller, an SDC representation of the above model is needed. Since the model has three state variables, there are infinite ways to form the state-dependent matrices. The following one is used in our implementation:

$$\dot{x}(t) = F(x(t))x(t) + bu(t), \quad y(t) = Hx(t) \tag{18}$$

where $x(t) = [h_1(t)\ h_3(t)\ h_2(t)]^T$, and

$$F(x(t)) = \begin{bmatrix} F_{11}(x(t)) & F_{12}(x(t)) & 0 \\ F_{21}(x(t)) & F_{22}(x(t)) & F_{23}(x(t)) \\ 0 & F_{32}(x(t)) & F_{33}(x(t)) \end{bmatrix}$$

$$b = \begin{bmatrix} \dfrac{1}{A_t} & 0 \\ 0 & 0 \\ 0 & \dfrac{1}{A_t} \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

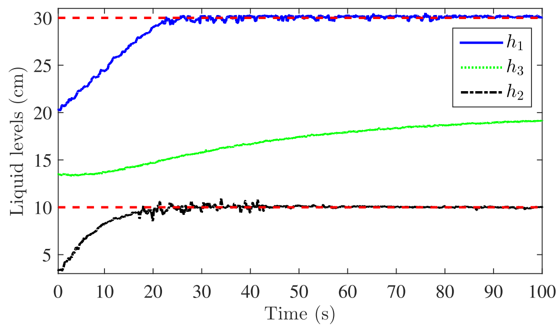

Fig. 5. Graphs of the levels of the tanks. Dashed lines: desired trajectories.

The elements of $F(x(t))$ are as follows:

$$F_{12}(x(t)) = -F_{11}(x(t)) = \frac{a_1 S\sqrt{2g}}{A_t}(h_1(t) - h_3(t))^{\frac{-1}{2}}$$
$$F_{21}(x(t)) = \frac{a_1 S\sqrt{2g}}{A_t}(h_1(t) - h_3(t))^{\frac{-1}{2}}$$
$$F_{23}(x(t)) = \frac{a_3 S\sqrt{2g}}{A_t}(h_3(t) - h_2(t))^{\frac{-1}{2}}$$
$$F_{22}(x(t)) = -F_{21}(x(t)) - F_{23}(x(t))$$
$$F_{32}(x(t)) = \frac{T a_3 S\sqrt{2g}}{A_t}(h_2(t) - h_3(t))^{\frac{-1}{2}}$$
$$F_{33}(x(t)) = -F_{32} - \frac{a_2 S\sqrt{2g}}{A_t}(h_2(t))^{\frac{-1}{2}}.$$

As it was mentioned above, the problem is to set the levels of the first and second tanks to the desired constant values. Therefore, the dynamics $\dot{x}_d(t) = 0_{2\times 1}$ and $y_d(t) = x_d(t)$ is used to describe the desired trajectory.

Due to the assumption $h_1(t) > h_3(t) > h_2(t)$, one can see that the above SDC representation (18) is pointwise controllable. On the other hand, by selecting the discounted factor $\gamma = 0.01$ and the weighting parameters $Q_1 = 100I_2$ and $R = 0.05I_2$ and using Theorems 1 and 2, it is possible to show that the closed-loop system is stable and the tracking error converges to zero. Fig. 5 shows the levels of the tanks for $a_i = 0.5$, $S = 0.5$, $A_t = 0.0154$, and $g = 9.81$. From this figure, it can be seen that the proposed method is so effective and the levels of the tanks are successfully set to their desired values.

Remark 2: Finding the solution of the SDRE (8) is the central component of the proposed tracking controller. While this equation might be solved analytically, a sampled-data method, represented in [8], is used in this brief.

IV. Conclusion

Using a discounted cost function, a general optimal tracking problem has been considered for a broad class of nonlinear systems. The tracking problem has been converted into an optimal regulation problem without any discount factor by defining some new state variables and control input. In order to avoid encountering any HJB equations, the SDRE technique has been used to find a suboptimal solution of the obtained regulation problem. It has been shown that this control law has actually a feedback-feedforward structure for the original tracking problem, where both the feedback and feedforward gains are calculated by solving a state-dependent algebraic Riccati equation. The proposed method has been systematically applied to the problem of dive plane control of an AUV. Simulation results show that the proposed method is so effective to control nonlinear systems even in the presence of input saturation and parametric uncertainties. Capabilities of the proposed tracking controller have also been evaluated using an experimental three-tank system.

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