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$\mathcal{H}_\infty$ Suboptimal Tracking Controller Design for a Class of Nonlinear Systems

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Abstract: In this paper, a new technique is proposed to solve the $\mathcal{H}_\infty$ tracking problem for a broad class of nonlinear systems. Towards this end, based on a discounted cost function, a nonlinear two-player zero-sum differential (NTPZSD) game is defined. Then, the problem is converted to another NTPZSD game without any discount factor in its corresponding cost function. A state-dependent Riccati equation (SDRE) technique is applied to the latter NTPZSD game in order to find its approximate solution which leads to obtain a feedback-feedforward control law for the original game. It is proved that the tracking error between the system state and its desired trajectory converges asymptotically to zero under mild conditions on the discount factor. The proposed $\mathcal{H}_\infty$ tracking controller is applied to two nonlinear systems (the Vander Pol's oscillator and the insulin-glucose regulatory system of type I diabetic patients). Simulation results demonstrate that the proposed $\mathcal{H}_\infty$ tracking controller is so effective to solve the problem of tracking time-varying desired trajectories in nonlinear dynamical systems.

Keywords: $\mathcal{H}_\infty$ tracking controller, nonlinear two-player zero-sum differential (NTPZSD) game, time-varying desired trajectory, state-dependent Riccati equation (SDRE).

1. INTRODUCTION

The $\mathcal{H}_\infty$ control theory became one of the the most significant accomplishments in automatic control theory due to its effectiveness of attenuating the effect of disturbances in dynamical systems. The $\mathcal{H}_\infty$ control has a close connection with the two-player zero-sum differential games where one player tries to minimize a predefined cost function while the other tries to maximize it [1]. In many practical engineering problems, time-varying desired trajectories should be tracked by the system. While the $\mathcal{H}_\infty$ regulation problems in linear and nonlinear systems were studied by many researchers and considerable results were obtained (see e.g., [2–5]), less attention were paid to the $\mathcal{H}_\infty$ trajectory tracking problem (see e.g., [6,7]). Some reasons of this shortage are as follows: (i) Solving the $\mathcal{H}_\infty$ tracking problem leads to a complex Hamilton-Jacobi-Isaac (HJI) equation, which is too difficult or even impossible to be solved; (ii) There is an additional computational complexity caused by computing a feedforward term which is not presented in the $\mathcal{H}_\infty$ regulation problem.

The state-dependent Riccati equation (SDRE) technique, which was originally proposed by Pearson in 1962 [8], was systematically developed to solve many different control engineering problems in nonlinear dynamical systems such as robust $\mathcal{H}_\infty$ filter design [9], observer design for nonlinear delayed systems [10], sliding mode control design for delayed systems [11], and trajectory tracking design [12,13]. The key idea behind an SDRE technique is in representing the nonlinear system dynamics as a state-dependent linear system which is called the pseudo linearization, extended linearization, and/or state-dependent coefficient (SDC) matrix representation [14]. The SDRE based methods were also applied to many different problems in a wide variety of applications such as drug administration in cancer treatment [15] and rigid and flexible joint manipulator control [16]. Some reported reasons for this popularity are as follows [17]: (i) The SDRE techniques are based on simple concepts directly inherited from the well-established linear theories; (ii) The SDRE techniques preserve the nonlinearities of the system without neglecting any nonlinear terms; (iii) By selecting two weighting matrices, the overal system performance can be directly affected with predictable outcomes. Two comprehensive surveys of the SDRE techniques and the related theories are [14] and [18].

To benefit from the above mentioned advantages of the SDRE technique, a new SDRE based method is proposed to solve the $\mathcal{H}_\infty$ tracking problem in nonlinear dynamical systems. Towards this end, a general $\mathcal{H}_\infty$ tracking problem is firstly defined. Then, using an augmented system of the tracking error dynamics and the command generator dynamics, the defined $\mathcal{H}_\infty$ tracking problem is converted to another one. The latter problem is next solved by applying the SDRE technique. It is proved that the tracking error between the system state variable and its desired tra-

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jectory converges asymptotically to zero under mild conditions. The developed $\mathcal{H}_\infty$ tracking controller inherits almost all of the above mentioned interesting properties of the SDRE technique. Two numerical simulations (the Vander Pol's oscillator and the insulin-glucose regulatory system of type I diabetic patients) are worked to evaluate the capabilities of the proposed $\mathcal{H}_\infty$ tracking controller.

The remainder of the paper is organized as follows: In Section 2, an $\mathcal{H}_\infty$ nonlinear tracking problem is defined for a broad class of nonlinear dynamical systems. In Section 3, using the pseudo linearization technique, a method is proposed to find an approximate solution of the $\mathcal{H}_\infty$ tracking problem. Asymptotic stability of the closed-loop system under the control law obtained from the proposed $\mathcal{H}_\infty$ tracking controller is also investigated in this section. In Section 4, simulation results of applying the proposed method to two nonlinear trajectory tracking problems (the Vander Pol's oscillator and glucose level control of type I diabetic patients) are presented. Finally, Section 5 concludes the paper.

2. SYSTEM DESCRIPTION AND PROBLEM STATEMENT

Consider the following nonlinear dynamical system described by:

$$\begin{aligned}\dot{x}(t) &= f(x(t)) + b_1(x(t))u(t) + b_2(x(t))d(t),\\ x(0) &= x_0,\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $d(t) \in \mathbb{R}^q$ is the external disturbance, and x_0 is the initial condition. $f(x(t)) : \mathbb{R}^n \to \mathbb{R}^n$, $b_1(x(t)) : \mathbb{R}^n \to \mathbb{R}^m$, and $b_2(x(t)) : \mathbb{R}^n \to \mathbb{R}^q$ are assumed to be smooth functions, $f(0) = 0$, and $b_1(x(t)) \neq 0$ for all $x(t) \in \mathbb{R}^n$.

The $\mathcal{H}_\infty$ tracking problem is to find the control input $u(t)$, $t \geq 0$ such that the system state $x(t)$, $t \geq 0$ asymptotically tracks the desired trajectory $x_d(t)$, $t \geq 0$ and the effects of the disturbance input $d(t)$ on the system performance are attenuated. In this paper, the following definition is used to mathematically define the $\mathcal{H}_\infty$ tracking problem. Note that this $\mathcal{H}_\infty$ tracking problem can also be used when the desired trajectories are not generated by an asymptotically stable system.

$\mathcal{H}_\infty$ nonlinear tracking problem: For the nonlinear system (1), find the control input $u(t)$, $t \geq 0$ such that the tracking error $e(t) \triangleq x(t) - x_d(t)$, $t \geq 0$ tends to zero as t tends to infinity with $d(t) = 0$, $t \geq 0$ and it has L_2-gain less than or equal to $\gamma > 0$, that is

$$\int_0^\infty e^{-2\alpha t}\big(e^{\mathrm{T}}(t)Qe(t) + u^{\mathrm{T}}(t)Ru(t)\big)\mathrm{d}t \leq \gamma^2 \int_0^\infty e^{-2\alpha t} d^{\mathrm{T}}(t)d(t)\mathrm{d}t$$

for all $d(t) \in L_2[0,\infty)$, where $\alpha > 0$ is the discount factor, Q and R are respectively positive-semidefinite and positive-definite symmetric matrices with appropriate dimensions.

It is well-known that the $\mathcal{H}_\infty$ control problem has a close conection with two-player zero-sum differential games [4]. Indeed, the solution of the above $\mathcal{H}_\infty$ tracking problem can be found by solving the saddle point of its equivalent two-player zero-sum differential game. Therefore, the following min-max optimization problem is defined based on the above $\mathcal{H}_\infty$ tracking problem. In this way, we call $u(t)$ a minimizing player and $d(t)$ a maximizing player.

Nonlinear two-player zero-sum differential (NTPZSD) game: For the nonlinear system (1), find the control input $u(t) = u^\star(t)$, $t \geq 0$ that minimizes the following cost function and the disturbance $d(t) = d^\star(t)$, $t \geq 0$ that maximizes the cost function.

$$\begin{aligned}J(e_0, u(t), d(t)) = \int_0^\infty e^{-2\alpha t}\big(&e^{\mathrm{T}}(t)Qe(t)\\ &+ u^{\mathrm{T}}(t)Ru(t) - \gamma^2 d^{\mathrm{T}}(t)d(t)\big)\mathrm{d}t\end{aligned} \tag{2}$$

Applying the Bellman's principle of optimality to this NTPZSD game leads to a complicated HJI equation which is too difficult or even impossible to be solved. In the following section, based on the pseudo linearization idea, a new method is proposed to find approximate solutions of this problem.

3. CONTROLLER DESIGN METHODOLOGY

Assume the desired trajectory $x_d(t)$ has the following nonlinear dynamics:

$$\begin{aligned}\dot{z}_d(t) &= f_d(z_d(t)), \quad z_d(0) = z_{d0},\\ x_d(t) &= h_d(z_d(t)),\end{aligned} \tag{3}$$

where $z_d(t) \in \mathbb{R}^{n_d}$ and $x_d(t) \in \mathbb{R}^n$ are the state and the output of the desired trajectory system, respectively, and z_{d0} is the initial condition. Functions $f_d(z_d(t)) : \mathbb{R}^{n_d} \to \mathbb{R}^{n_d}$ and $h_d(z_d(t)) : \mathbb{R}^{n_d} \to \mathbb{R}^n$ are assumed to be smooth and $f_d(0) = h_d(0) = 0$.

Remark 1: Many useful desired trajectories such as steps, sinusoidal signals, and damped sinusoids can be generated by (3).

Defining the tracking error $e(t) \triangleq x(t) - x_d(t)$ and using (1) and (3), the tracking error dynamics is obtained as follows:

$$\begin{aligned}\dot{e}(t) =& f(x(t)) + b_1(x(t))u(t) + b_2(x(t))d(t)\\ &- \frac{\partial h_d(z_d(t))}{\partial(z_d(t))} f_d(z_d(t)).\end{aligned} \tag{4}$$

Since $f(x(t))$, $f_d(x_d(t))$, and $h_d(z_d(t))$ are smooth and $f(0) = f_d(0) = h_d(0) = 0$, these functions and $\partial h_d(z_d(t))/\partial(z_d(t)) f_d(z_d(t))$ can be rewritten in their

SDC representations as follows [18]:

$$\begin{aligned} &f(x(t)) = F(x(t))x(t), \\ &f_{\mathrm{d}}(z_{\mathrm{d}}(t)) = F_{\mathrm{d}}(z_{\mathrm{d}}(t))z_{\mathrm{d}}(t), \\ &h_{\mathrm{d}}(z_{\mathrm{d}}(t)) = H_{\mathrm{d}}(z_{\mathrm{d}}(t))z_{\mathrm{d}}(t), \\ &\frac{\partial h_{\mathrm{d}}(z_{\mathrm{d}}(t))}{\partial (z_{\mathrm{d}}(t))} f_{\mathrm{d}}(z_{\mathrm{d}}(t)) = G_{\mathrm{d}}(z_{\mathrm{d}}(t))z_{\mathrm{d}}(t), \end{aligned} \tag{5}$$

where $F(x(t)) : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $F_{\mathrm{d}}(z_{\mathrm{d}}(t)) : \mathbb{R}^{n_{\mathrm{d}}} \to \mathbb{R}^{n_{\mathrm{d}}\times n_{\mathrm{d}}}$, $H_{\mathrm{d}}(z_{\mathrm{d}}(t)) : \mathbb{R}^{n_{\mathrm{d}}} \to \mathbb{R}^{n\times n_{\mathrm{d}}}$, and $G_{\mathrm{d}}(z_{\mathrm{d}}(t)) : \mathbb{R}^{n_{\mathrm{d}}} \to \mathbb{R}^{n\times n_{\mathrm{d}}}$ are four matrix-valued functions. Note that the non-unique SDC matrix representations of the above functions provides an extra design degree of freedom which could be utilized to enhance the overal system performance [15].

Now, define $X(t) \triangleq e^{-\alpha t}\begin{bmatrix} e^{\mathrm{T}}(t) & z_{\mathrm{d}}^{\mathrm{T}}(t)\end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{n+n_{\mathrm{d}}}$, $U(t) \triangleq e^{-\alpha t}u(t) \in \mathbb{R}^m$, and $D(t) \triangleq e^{-\alpha t}d(t) \in \mathbb{R}^q$. Substituting them in (2) leads to the following cost function:

$$\begin{aligned} J(X_0, U(t), D(t)) = \int_0^\infty \big(&X^{\mathrm{T}}(t)\tilde{Q}X(t) + U^{\mathrm{T}}(t)RU(t) \\ &- \gamma^2 D^{\mathrm{T}}(t)D(t)\big)\mathrm{d}t, \end{aligned} \tag{6}$$

where

$$\tilde{Q} = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix},$$

and 0 denotes the zero matrix with appropriate dimensions. On the other hand, by substituting $\dot{z}_{\mathrm{d}}(t)$ and $\dot{e}(t)$ from (3) and (4), respectively, and using (5), the following SDC representation is obtained for the augmented state variable $X(t)$:

$$\begin{aligned} \dot{X}(t) =& A(e^{\alpha t}X(t))X(t) + B_1(e^{\alpha t}X(t))U(t) \\ &+ B_2(e^{\alpha t}X(t))D(t), \end{aligned} \tag{7}$$

where

$$\begin{aligned} A(e^{\alpha t}X(t)) &= -\alpha I \\ &\quad + \begin{bmatrix} F(x(t)) & F(x(t))H_{\mathrm{d}}(z_{\mathrm{d}}(t)) - G_{\mathrm{d}}(z_{\mathrm{d}}(t)) \\ 0 & F_{\mathrm{d}}(x_{\mathrm{d}}(t)) \end{bmatrix}, \\ B_1(e^{\alpha t}X(t)) &= \begin{bmatrix} b_1^{\mathrm{T}}(x(t)) & 0 \end{bmatrix}^{\mathrm{T}}, \\ B_2(e^{\alpha t}X(t)) &= \begin{bmatrix} b_2^{\mathrm{T}}(x(t)) & 0 \end{bmatrix}^{\mathrm{T}}, \end{aligned}$$

and I denotes the identity matrix with appropriate dimensions.

Remark 2: It is also possible to write the dynamics of the new state variable $X(t)$ as $\dot{X}(t) = g(e^{\alpha t}X(t)) + B_1(e^{\alpha t}X(t))U(t) + B_2(e^{\alpha t}X(t))D(t)$ and then find an SDC representation for this dynamics.

For simplicity in the notation, the argument t in $x(t)$, $x_{\mathrm{d}}(t)$, $X(t)$, $U(t)$, and $D(t)$ is omitted in some places in the paper. To sum up, it can be said that the NTPZSD game (1) and (2) is converted to another NTPZSD game described by (6) and (7). The minimizing control and the worst case disturbance of the latter NTPZSD game are as follows [5]:

$$\begin{aligned} U^\star(t) &= -\frac{1}{2}B_1^{\mathrm{T}}(e^{\alpha t}X(t))\frac{\partial V^\star(t, X(t))}{\partial X(t)}, \\ D^\star(t) &= \frac{1}{2\gamma^2}B_2^{\mathrm{T}}(e^{\alpha t}X(t))\frac{\partial V^\star(t, X(t))}{\partial X(t)}, \end{aligned}$$

where $V^\star(t, X(t))$ is the solution of the following HJI equation:

$$\begin{aligned} \min_{U\in L_2[0,\infty)} \max_{D\in L_2[0,\infty)} \Big\{ &\frac{\partial V^\star(t,X)}{\partial X}\big(A(e^{\alpha t}X)X + B_1(e^{\alpha t}X)U \\ &+ B_2(e^{\alpha t}X)D\big) + X^{\mathrm{T}}\tilde{Q}X + U^{\mathrm{T}}RU - \gamma^2 D^{\mathrm{T}}D\Big\} = 0. \end{aligned} \tag{8}$$

Unfortunately, solving the above complicated HJI equation is not generally easier than the HJI equation arisen from the original NTPZSD game. Therefore, developing a systematic method to find approximate solutions of the problem is in our interest. As a powerful alternative to the HJI technique, the SDRE technique provides very effective algorithms for synthesizing the nonlinear control laws. Therefore, in the rest of this section, the SDRE technique is applied to the NTPZSD game (6) and (7). Let us first present some necessary definitions which are needed in the rest of the paper. Hereafter, $\Omega \in \mathbb{R}^{2n}$ is a bounded open set containing the origin.

Definition 1: The SDC representation (7) is pointwise stabilizable in Ω if the pair $\big(A(e^{\alpha t}X(t)), B_1(e^{\gamma t}X(t))\big)$ is stabilizable for all $X(t) \in \Omega$ and $t \geq 0$.

Definition 2: The SDC representation (7) is pointwise detectable in Ω if the pair $\big(A(e^{\alpha t}X(t)), \tilde{Q}^{1/2}\big)$ is detectable for all $X(t) \in \Omega$ and $t \geq 0$.

Now, an SDRE technique, proposed in [19], is used to find a suboptimal solution of the NTPZSD game (6) and (7). Towards this end, two steps below must be taken [19].

1) Find the symmetric positive-definite matrix $P(e^{\alpha t}X)$ from the following SDRE.

$$\begin{aligned} &A^{\mathrm{T}}(e^{\alpha t}X)P(e^{\alpha t}X) + P(e^{\alpha t}X)A(e^{\alpha t}X) \\ &- P(e^{\alpha t}X)B_1(e^{\alpha t}X)R^{-1}B_1^{\mathrm{T}}(e^{\alpha t}X)P(e^{\alpha t}X) \\ &+ \frac{1}{\gamma^2}P(e^{\alpha t}X)B_2(e^{\alpha t}X)B_2^{\mathrm{T}}(e^{\alpha t}X)P(e^{\alpha t}X) + \tilde{Q} = 0. \end{aligned} \tag{9}$$

2) Compute the control law $U^\star(t)$ and the worst case disturbance $D^\star(t)$ via

$$U^\star(t) = -R^{-1}B_1^{\mathrm{T}}(e^{\alpha t}X(t))P(e^{\alpha t}X(t))X(t), \tag{10}$$

$$D^\star(t) = \frac{1}{\gamma^2}B_2^{\mathrm{T}}(e^{\alpha t}X(t))P(e^{\alpha t}X(t))X(t). \tag{11}$$

Note that if the triple $\big(A(e^{\alpha t}X(t)), B_1(e^{\alpha t}X(t)), \tilde{Q}^{1/2}\big)$ is point-wise stabilizable and point-wise detectable, the

SDRE (9) has a unique symmetric positive-definite solution $P(e^{\alpha t}X(t))$ for sufficiently large values of γ [19]. As one can see, instead of the HJI equation (8), the SDRE (9) is solved to find the solution of the NTPZSD game (6) and (7). It should be noted that this SDRE can be solved analytically or numerically [14]. The following theorem shows that under which conditions the control law (10) stabilizes the nonlinear system (7).

Theorem 1: Consider the nonlinear dynamical system (7). Assume the triple $\big(A(e^{\alpha t}X(t)), B_1(e^{\alpha t}X(t)), \tilde{Q}^{1/2}\big)$ is point-wise stabilizable and point-wise detectable in $\Omega = \Omega_x \times \Omega_{z_d}$. For $D(t) = 0, t \geq 0$, sufficiently large values of γ, and $\alpha \to 0$, the origin of the system (7) under the control law (10) is locally asymptotically stable, where $P(e^{\alpha t}X(t))$ is the solution of the SDRE (9). Furthermore, the tracking error $e(t)$ converges to zero as t tends to infinity.

Proof: For $\alpha \to 0$ and sufficiently large values of γ and due to the point-wise stabilizability and point-wise detectability of the triple $\big(A(e^{\alpha t}X(t)), B_1(e^{\alpha t}X(t)), \tilde{Q}^{1/2}\big)$, the state-dependent Riccati equation (9) has a unique symmetric positive-definite solution $P(e^{\alpha t}X(t))$ [19]. Using Taylor series expansion, it is possible to write $A(e^{\alpha t}X(t))$, $B_1(e^{\alpha t}X(t))$, and $P(e^{\alpha t}X(t))$ as follows:

$$A(X(t)) = A^0 + \Delta A(X(t)),$$
$$B_1(X(t)) = B_1^0 + \Delta B_1(X(t)),$$
$$P(X(t)) = P^0 + \Delta P(X(t)),$$

where $A^0 = A(0)$, $B_1^0 = B_1(0)$, and $P^0 = P(0)$; $\Delta A(X(t))$, $\Delta B_1(X(t))$, and $\Delta P(X(t))$ are higher order terms of Taylor series expansions of $A(X(t))$, $B_1(X(t))$, and $P(X(t))$, respectively. The nonlinear dynamics (7) under the control law (10) can be written as follows:

$$\begin{aligned}\dot{X}(t) =& A(X(t))X(t) - B_1(X(t))R^{-1}B_1^{\mathrm{T}}(X(t)) \\ & \times P(X(t))X(t) \triangleq A_{\mathrm{cl}}(X(t))X(t).\end{aligned} \tag{12}$$

On the other hand, it is possible to rewrite $A_{\mathrm{cl}}(X(t))$ as follows:

$$A_{\mathrm{cl}}(X(t)) = A_{\mathrm{cl}}^0 + \Delta A_{\mathrm{cl}}(X(t)),$$

where

$$A_{\mathrm{cl}}^0 = A^0 - B_1^0 R^{-1}(B_1^0)^{\mathrm{T}} P^0. \tag{13}$$

For small values of $X(t)$ and $\alpha \to 0$, $\Delta A(X(t))$, $\Delta B_1(X(t))$, and $\Delta P(X(t))$ tend to zero and therefore, $\Delta A_{\mathrm{cl}}(X(t))$ also tends to zero. Now, consider the Lyapunov function $V(X(t)) = X^{\mathrm{T}}(t)P^0X(t)$. Using (12), the derivative of this Lyapunov function is given as:

$$\begin{aligned}\dot{V}(X(t)) &= X^{\mathrm{T}}(t)\big(A_{\mathrm{cl}}^{\mathrm{T}}(X(t))P^0 + P^0A_{\mathrm{cl}}(X(t))\big)X(t) \\ &= X^{\mathrm{T}}(t)\big((A_{\mathrm{cl}}^0)^{\mathrm{T}}P^0 + P^0A_{\mathrm{cl}}^0 + \sigma(X(t))\big)X(t),\end{aligned}$$

where

$$\sigma(X(t)) = (\Delta A_{\mathrm{cl}}(X(t)))^{\mathrm{T}}P^0 + P^0\Delta A_{\mathrm{cl}}(X(t)).$$

Using (13), the above equality can be written as follows:

$$\begin{aligned}\dot{V}(X(t)) = &- X^{\mathrm{T}}(t)\big(\tilde{Q} + P^0B_1^0R^{-1}(B_1^0)^{\mathrm{T}}P^0 \\ &- \frac{1}{\gamma^2}P^0B_2(X)B_2^{\mathrm{T}}(X)P^0\big)X(t) \\ &+ X^{\mathrm{T}}(t)\sigma(X(t))X(t).\end{aligned}$$

On the other hand, since $\Delta A_{\mathrm{cl}}(X(t))$ tends to zero for small values of $X(t)$ and $\alpha \to 0$, the second term in the right hand side of the above equality can be neglected and thus:

$$\begin{aligned}\dot{V}(X(t)) = &- X^{\mathrm{T}}(t)\big(\tilde{Q} + P^0B_1^0R^{-1}(B_1^0)^{\mathrm{T}}P^0 \\ &- \frac{1}{\gamma^2}P^0B_2(X(t))B_2^{\mathrm{T}}(X(t))P^0\big)X(t).\end{aligned}$$

For sufficiently large values of γ and since $P^0 > 0$, one can conclude that $\dot{V}(X(t)) < 0$ and hence, the augmented state $X(t) = e^{-\alpha t}\big[e^{\mathrm{T}}(t) \quad x_{\mathrm{d}}^{\mathrm{T}}(t)\big]^{\mathrm{T}}$ asymptotically tends to zero. On the other hand, since $\alpha \to 0$, it can also be concluded that the tracking error $e(t)$ tends to zero. The proof is completed. □

Remark 3: As it is proved, to guarantee the local asymptotic stability of the error $e(t)$, the discount factor α should be a small positive number. On the other hand, based on Theorem 2 in [20], to have a point-wise stabilizable pair $\big(A(e^{\alpha t}X(t)), B_1(e^{\alpha t}X(t))\big)$ (which is necessary to have the positive-definite solution of the SDRE (9)), α must be greater than the real parts of the eigenvalues of $F_{\mathrm{d}}(z_{\mathrm{d}}(t))$ for all $z_{\mathrm{d}}(t) \in \Omega_{z_{\mathrm{d}}}$. Therefore, the discount factor α is a critical parameter of the proposed $\mathcal{H}_\infty$ tracking controller which should be selected based on these two conditions.

From (10) and (11), the following feedback-feedforward control law and the worst case disturbance are obtained for the original NTPZSD game (1) and (2):

$$u^\star(x, x_{\mathrm{d}}) = -R^{-1}B_1^{\mathrm{T}}(x)P(x, x_{\mathrm{d}})\big[x^{\mathrm{T}} \quad x_{\mathrm{d}}^{\mathrm{T}}\big]^{\mathrm{T}},$$
$$d^\star(x, x_{\mathrm{d}}) = \frac{1}{\gamma^2}B_2^{\mathrm{T}}(x)P(x, x_{\mathrm{d}})\big[x^{\mathrm{T}} \quad x_{\mathrm{d}}^{\mathrm{T}}\big]^{\mathrm{T}},$$

which can be rewritten as follows:

$$u^\star(x, x_{\mathrm{d}}) = K_{f_1}(x, x_{\mathrm{d}})x - K_{ff_1}(x, x_{\mathrm{d}})x_{\mathrm{d}},$$
$$d^\star(x, x_{\mathrm{d}}) = K_{f_2}(x, x_{\mathrm{d}})x - K_{ff_2}(x, x_{\mathrm{d}})x_{\mathrm{d}},$$

where $K_{f_1}(x, x_{\mathrm{d}})$ and $K_{f_2}(x, x_{\mathrm{d}})$ are the feedback gains and $K_{ff_1}(x,\ x_{\mathrm{d}})$ and $K_{ff_2}(x, x_{\mathrm{d}})$ are the feedforward gains. It should be noted that both the feedback and the feedforward gains are simultaneously calculated by solving the SDRE (9).

4. SIMULATION RESULTS

In this section, the proposed $\mathcal{H}_\infty$ tracking controller is applied to two nonlinear dynamical systems. The first one is the Vander Pol's oscillator. The second example concerns the problem of the glucose level control of type I diabetic patients.

4.1. Vander Pol's oscillator

Consider the Vander Pol's oscillator

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) + d(t),\\ \dot{x}_2(t) &= -x_1(t) + \beta(1 - x_1^2(t))x_2(t) + u(t) + d(t).\end{aligned}$$

The objective is to design the control law $u(t)$ such that the state variables $x_1(t)$ and $x_2(t)$ track the desired trajectories $x_{d1}(t) = \sin(t)$ and $x_{d2}(t) = \cos(t)$, respectively, and the effects of the disturbance $d(t)$ on the system performance are minimized. The dynamics of the desired trajectory is as follows:

$$\begin{aligned}\dot{z}_d(t) &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} z_d(t), \quad z_d(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^{\mathrm{T}},\\ x_d(t) &= z_d(t).\end{aligned}$$

To apply the proposed $\mathcal{H}_\infty$ tracking controller, there are an infinite number of ways to find the SDC representation of the augmented state $X(t) = e^{-\alpha t}\begin{bmatrix} x_1(t) - x_{d1}(t) & x_2(t) - x_{d2}(t) & x_{d1}(t) & x_{d2}(t) \end{bmatrix}^{\mathrm{T}}$. The following one is used in our simulations:

$$\begin{aligned}\dot{X}(t) = &\begin{bmatrix} -\alpha & 1 & 0 & 0 \\ -1 & -\alpha + a_{24}(x_1) & 0 & a_{24}(x_1) \\ 0 & 0 & -\alpha & 1 \\ 0 & 0 & -1 & -\alpha \end{bmatrix} X(t)\\ &+ B_1 U(t) + B_2 D(t),\end{aligned} \tag{14}$$

where $a_{24}(x_1) = \beta(1 - x_1^2)$, $B_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$, and $B_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$. Paying attention to Theorem 1, to apply the proposed $\mathcal{H}_\infty$ tracking controller to the above problem, the point-wise stabilizability of the SDC representation (14) must be checked. One can see that the state-dependent controllability matrix of the SDC representation (14) has rank 2 for all $X(t)$ and the first two state variables are point-wise controllable while the second ones are not. Nonetheless, these state variables are stabilizable and hence, using the proposed $\mathcal{H}_\infty$ tracking controller leads to a stable closed-loop system for $\alpha \to 0$. The simulation is done for $d(t) = 0.5e^{-t}$, $t \geq 0$, $Q = 10I$, $R = 1$, $\gamma = 5$, $\alpha = 0.1$, $\beta = 0.9$, and $X_0 = \begin{bmatrix} 2 & -2 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$. The obtained system state variables $x_1(t)$ and $x_2(t)$ and their desired values are depicted in Figs. 1 and 2, respectively. The corresponding control input $u(t)$ is shown in Fig. 3. From these figures, it can be concluded that the results are satisfactory and the tracking goal is successfully achieved.

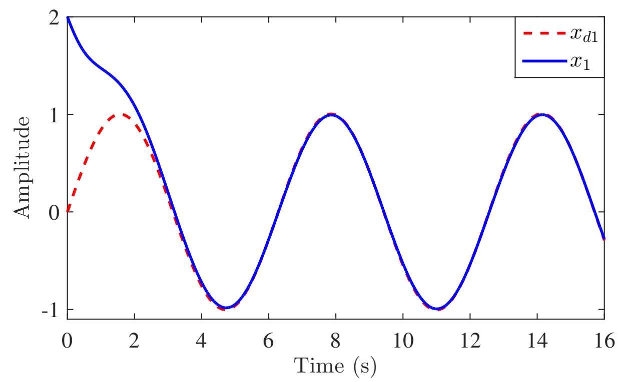

Fig. 1. The system state $x_1(t)$ and its desired $x_{d1}(t)$.

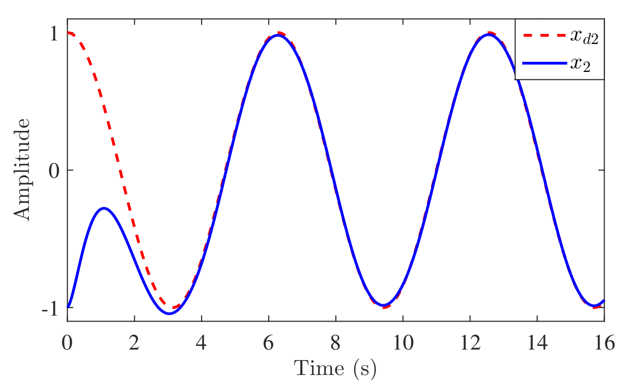

Fig. 2. The system state $x_2(t)$ and its desired $x_{d2}(t)$.

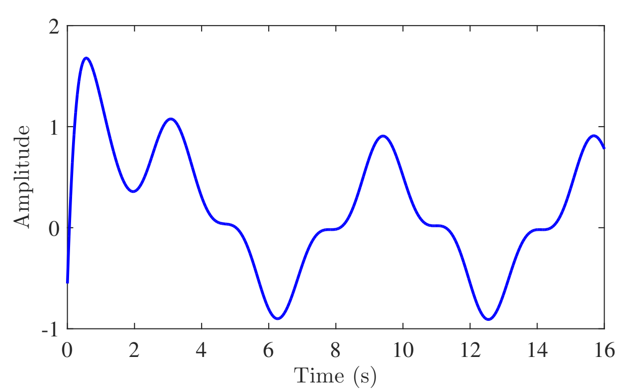

Fig. 3. The corresponding control input for the Vander Pol's oscillator.

Remark 4: From the above example, it can be seen that the control input $u^\star(t)$ is applied to the system even if the disturbance $d(t)$ is not the same as the worst case $d^\star(t)$. Indeed, $u^\star(t)$ is obtained to minimize the cost function (2) by assuming that the system is in the presence of the worst case disturbance $d^{\star}(t)$ and therefore, the control is robust against the disturbance $d(t)$.

Remark 5: It should be noted that the smaller values of the parameter γ, the less effects of the disturbance $d(t)$ on the system performance. On the other hand, for too small values of γ, the SDRE (9) may not have a positive-definite solution. Therefore, it is obvious that a minimum value of γ is in our interest. However, there exists no way to find the smallest value of the parameter γ for general nonlinear systems, and a large enough value is usually predetermined for this parameter [6].

4.2. Glucose level control of type I diabetic patients

In this example, the minimal model of the insulin-glucose regulatory system is used to design insulin injection rules for type I diabetic (T1D) patients based on the proposed $\mathcal{H}_\infty$ tracking controller. The model consists of three components; $G(t)$: the glucose concentration in the blood plasma in mg/dl, $Y(t)$: the insulin concentration in the remote compartments in 1/min, and I: the insulin concentration in the blood plasma in μU/ml. Interactions of each compartment with the others are given by the following ordinary differential equations [21]:

$$\begin{aligned}
\dot{G}(t) &= -p_1(G(t)-G_\mathrm{b}) - G(t)Y(t) + d(t),\\
\dot{Y}(t) &= -p_2 Y(t) + p_3(I(t)-I_\mathrm{b}),\\
\dot{I}(t) &= -n(I(t)-I_\mathrm{b}) + u(t),
\end{aligned} \tag{15}$$

where G_b and I_b are the basal glucose level and the basal insulin level, respectively, n is the time constant for insulin disappearance, p_1 is the insulin-independent rate constant of glucose uptake in muscles and liver, p_2 is the rate for decrease in tissue glucose uptake ability, p_3 is the insulin-dependent increase in glucose uptake ability in tissue per unit of insulin concentration above the basal level, $u(t)$ is the insulin injection rate, and $d(t)$ is the rate at which glucose is absorbed to the blood after food intake. The dynamical behaviour of $d(t)$ can be modeled by the following decaying exponential function [21]:

$$d(t) = a\exp(-bt), \tag{16}$$

where a and b are two positive constants. The parameter values of the model (15) for a specific diabetic patient are $G_\mathrm{b} = 90$ mg/dl, $I_\mathrm{b} = 7$ μU/ml, $n = 0.2814$ 1/min, $p_1 = 0$ 1/min, $p_2 = 0.0142$ 1/min, and $p_3 = 1.54\times10^{-5}$ ml/(μU×min) [22].

Since the term $d(t)$ is usually unknown with negative effects on the glucose level, hereafter it is considered as the disturbance signal. The control problem is to find the control law $u(t)$ such that the glucose concentration $G(t)$ tracks the desired constant value $G_\mathrm{d}(t) = G_\mathrm{b}$ and the effects of the disturbance $d(t)$ on the system performance are minimized. The dynamics of the desired trajectory is

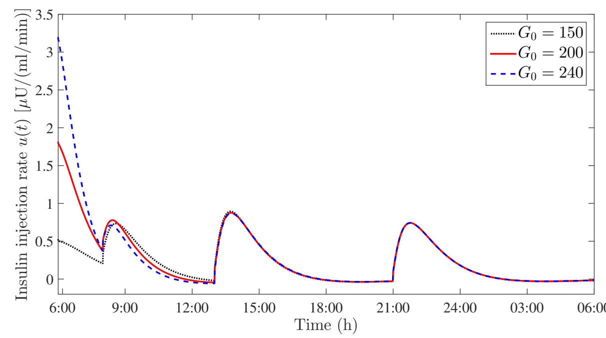

Fig. 4. Graphs of the insulin injection rate $u(t)$.

as follows:

$$\begin{aligned}
\dot{z}_\mathrm{d}(t) &= 0, \quad z_\mathrm{d}(0) = \begin{bmatrix} G_\mathrm{b} & 0 & I_\mathrm{b}\end{bmatrix}^\mathrm{T},\\
x_\mathrm{d}(t) &= z_\mathrm{d}(t).
\end{aligned} \tag{17}$$

To apply the proposed $\mathcal{H}_\infty$ tracking controller to this problem, the following SDC representation of the augmented state $X(t) = e^{-\alpha t}\begin{bmatrix} G(t) & Y(t) & I(t) & z_\mathrm{d}^\mathrm{T}(t)\end{bmatrix}^\mathrm{T}$ is considered based on (15) and (17):

$$\dot{X} = \begin{bmatrix} a_{11} & -\frac{G(t)}{2} & 0 & p_1 & 0 & 0\\ 0 & -\alpha-p_2 & p_3 & 0 & 0 & p_3\\ 0 & 0 & -\alpha-n & 0 & 0 & n\\ 0 & 0 & 0 & -\alpha & 0 & 0\\ 0 & 0 & 0 & 0 & -\alpha & 0\\ 0 & 0 & 0 & 0 & 0 & -\alpha\end{bmatrix} X + B_1U + B_2D,$$

where $a_{11} = -\alpha - p_1 - \frac{Y(t)}{2}$, $B_1 = \begin{bmatrix}0 & 0 & 1 & 0 & 0\end{bmatrix}^\mathrm{T}$, and $B_2 = \begin{bmatrix}1 & 0 & 0 & 0 & 0\end{bmatrix}^\mathrm{T}$. Based on the results of Theorem 1, the above SDC representation must be stabilizable for all $X(t)$. A simple calculation shows that the state-dependent controllability matrix of the above SDC representation has a minor of order 3 with determinant of $G(t)(p_3)^2/2$. Paying attention to the parameter values of the model (15) and since $G(t) > 0$, the state-dependent stabilizability condition of the above SDC representation is always satisfied for $\alpha > 0$ (note that the second three state variables, which are related to the desired trajectory, are stabilizable). Therefore, the proposed $\mathcal{H}_\infty$ tracking controller can be applied to the problem. The obtained insulin injection rates for $Q = 100I$, $R = 0.25$, $\gamma = 100$, $\alpha = 0.1$, and three different initial conditions are depicted in Fig. 4. The corresponding growth curves of the glucose concentration in the blood plasma are depicted in Fig. 5. In these simulations, it is assumed that the patient serves three meals during the day at 8:00 AM, 1:00 PM, and 9:00 PM. As it clearly appears from Fig. 5, the proposed $\mathcal{H}_\infty$ tracking controller provides treatment strategies which do not lead to any hypoglycemic conditions ($G < 70$) and the observed hyperglycemia ($G > 180$) is limited.

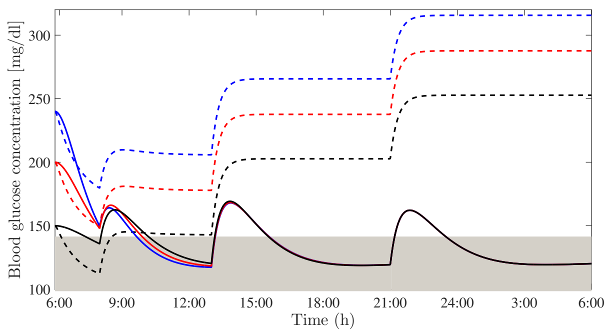

Fig. 5. Graphs of the blood glucose concentration with considering the effects of food intake: Solid lines are the blood glucose concentration trajectories controlled by the proposed $\mathcal{H}_\infty$ tracking controller; Dotted lines are the blood glucose concentration trajectories without any treatment; The gray band is the safe level.

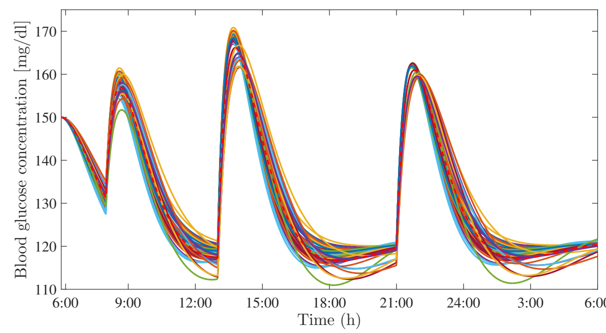

Fig. 6. Graphs of the blood glucose concentration for 50 T1D patients with unkonown parameters controlled by the proposed $\mathcal{H}_\infty$ tracking controller.

In the above $\mathcal{H}_\infty$ controller design, the parameter values of the model (15) must be known. The utilized values of these parameters are calculated for a patient of average weight [22]. However, these parameters are not constant numbers and vary from patient to patient. Therefore, the designed $\mathcal{H}_\infty$ tracking controller must exhibit robustness against the uncertainty in these parameters in order to maintain its performance. Having this property allows us to design the controller based on the nominal values of the parameters while the obtained insulin injection rate $u(t)$ is applied to the patient with different values of the parameters. In the following simulations, these parameters are considered to be random variables in the range of $\pm 40\%$ of their nominal values. It is also assumed that the patients consume three meals with unknown characteristics of meal disturbances, i.e., a and b in (16). The simulation results for 50 patients are depicted in Fig. 6. From this figure, it can be concluded that the $\mathcal{H}_\infty$ tracking controller is robust against parametric uncertainties, and the blood glucose concentration is successfully set into the safe level $\left[70 \;\; 140\right]$ mg/dl in a reasonable period of time.

5. CONCLUSION

Based on a discounted quadratic cost function, a nonlinear two-player zero-sum differential game has been considered in order to define an $\mathcal{H}_\infty$ tracking problem for a broad class of nonlinear systems. Using an augmented system of the tracking error dynamics and the command generator dynamics, the tracking problem has been converted to another nonlinear two-player zero-sum differential game without any discount factor where its control objective is to regulate the new state variable. An SDRE technique has been used to find an approximate solution of the latter problem or equivalently the original tracking problem. It has been shown that the obtained Nash equilibrium solution has a feedback-feedforward structure where both of the feedback and the feedforward gains are calculated by solving a state-dependent Riccati equation, simultaneously. It has been proved that the tracking error converges asymptotically to zero under a mild condition on the discount factor. Capabilities of the proposed $\mathcal{H}_\infty$ tracking controller have been evaluated using two numerical simulations.

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