Intelligent Control

Concept and Mathematics of Fuzzy Logic Systems

By:
Barmak Baigzadehnoe
b.baigzadeh@uok.ac.ir

Smart/Micro Grids Research Center, University of Kurdistan

Content

- Introduction to Fuzzy Systems
- Fuzzy Sets and Operations
- Linguistic Variables and Fuzzy Rules
- Fuzzy Implications
- Fuzzy Reasoning

Definition

*What is the meaning of the word "**Fuzzy**" ?*

- Not clear
- Indistinct
- Imprecise
- Vague
- Blurred
- Confused

Definition

Fuzzy Logic is a logic which refers to the study of methods and principles of human reasoning.

- ***Fuzzy Logic*** is a computational paradigm that is based on how humans think.
- ***Fuzzy Logic*** looks at the world in imprecise terms, in much the same way that our brain takes in information, then responds with precise actions.

Historical View

- Professor Lotfi Aliasker Zadeh (1965)
  - *First to publish ideas of fuzzy logic*
- Professor Toshire Terano (1972)
  - *Organized the world's first working group on fuzzy systems.*
- F.L. Smidth & Co. (1980)
  - *First to market fuzzy expert systems.*

Boolean Logic vs Fuzzy Logic

Slow

Fast

Boolean Logic

Slowest

Slow

Fast

Fastest

Fuzzy Logic

Fuzziness vs Probability

- ***Probability*** describes the uncertainty of an event occurrence
- ***Fuzziness*** describes event ambiguity

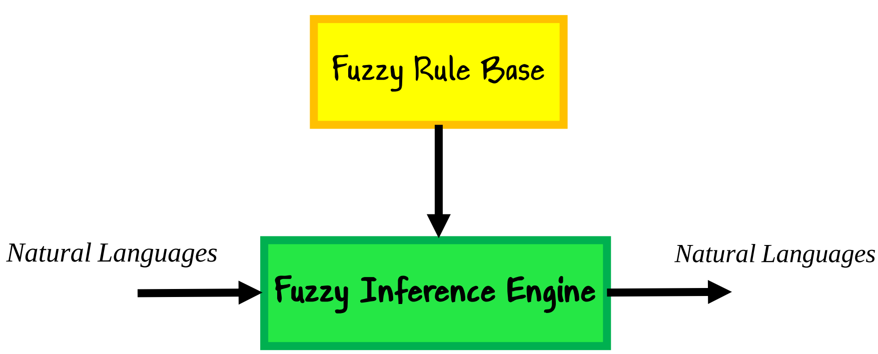
> Whether an event occurs is **RANDOM.**
> To what degree it occurs is **FUZZY.**

Example: Consider the following two statements:

- There is a 50% chance of an apple being in the refrigerator
- There is a half an apple in the refrigerator.

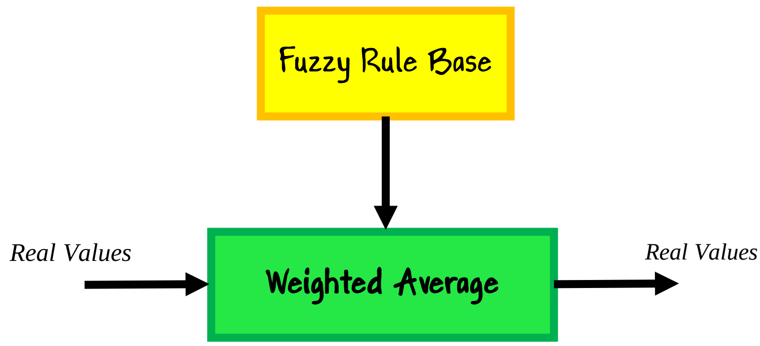
Fuzzy Logic Systems

- ***Pure Fuzzy System***



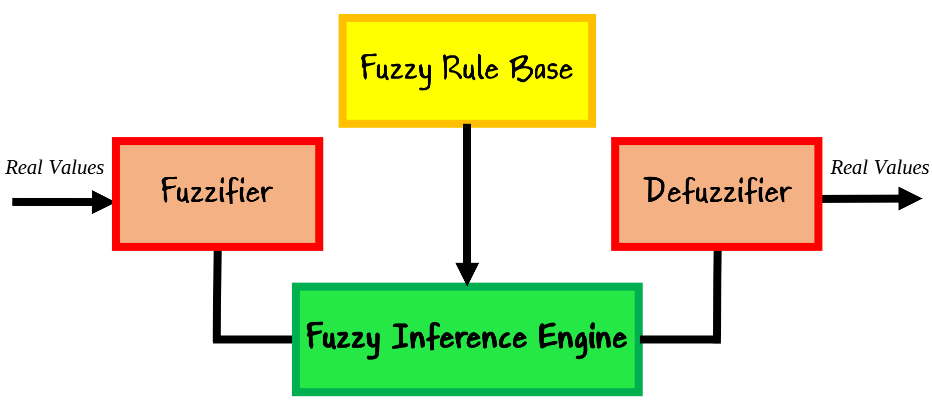
Fuzzy Logic Systems

- *Takagi-Sugeno-Kang Fuzzy System*



Fuzzy Logic Systems

- *Fuzzy System with Fuzzifier and Defuzzifier*



Why Fuzzy Logic Systems?

- Do not need exact and intricate mathematical models
- High accuracy and smooth control
- Simple to develop and implement
- Intrinsically Robust

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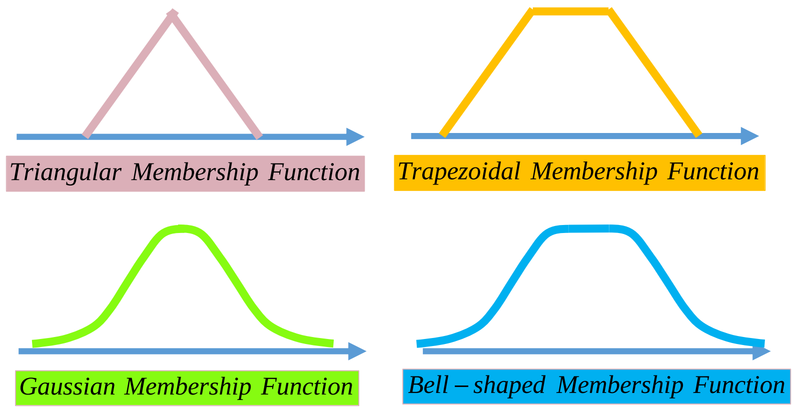
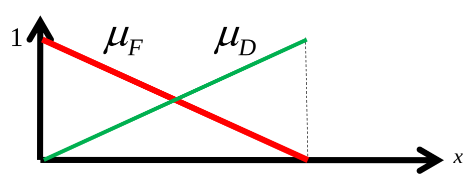
Membership Functions

Let A be a set from the domain X. A ***membership function*** of set A is a function that assigns value, or membership degree to every x ∈ A, μ: X → [0, 1].



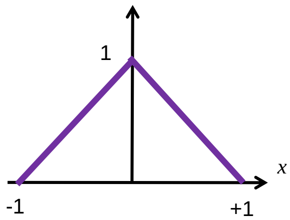
Membership Functions

Example: Fuzzy membership functions of Iranian and foreign cars:

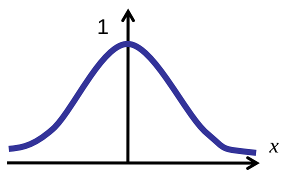


Membership Functions

Example: Fuzzy membership functions of numbers close to zero:



$$\mu_A(x)=\begin{cases}0 & x\le -1, x\ge 1\\ x+1 & -1\le x<0\\ 1-x & 0\le x<1\end{cases}$$

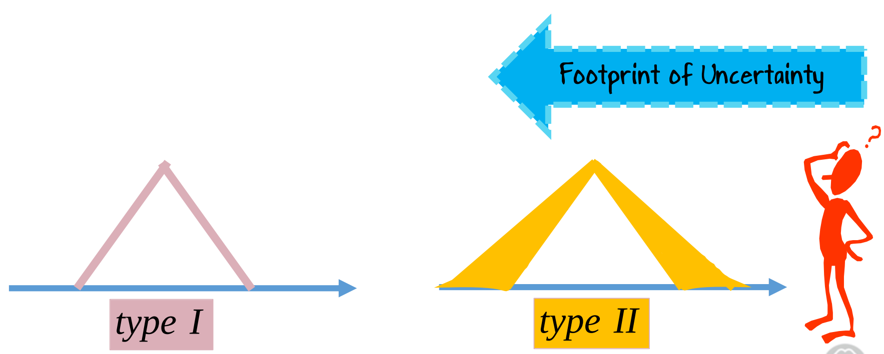


$$\mu_A(x)=e^{-x^2}$$

Membership Functions

Note: There are two well-known types of fuzzy membership functions:

- ***Type I membership Function***
- ***Type II membership Function***



Fuzzy Sets

A ***fuzzy set*** A in a universe of discourse U is characterized by a membership function $\mu_A(x)$ that takes values in the interval [0, 1]. Accordingly, a fuzzy set is represented as a set of ordered pairs :

$$A = \{(x, \mu_A(x)) \,|\, x \in X\}$$

Note: fuzzy sets commonly written as:

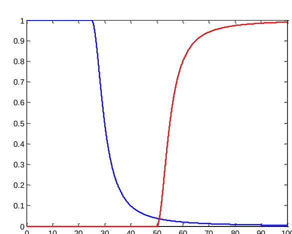
- Continuous: $A = \int_U \frac{\mu_A(x)}{x}$
- Discrete: $A = \sum_U \frac{\mu_A(x)}{x}$

Fuzzy Sets

Example: Let U be the interval [0,100] representing the age of ordinary humans. Then we may define fuzzy sets "young" and "old" as:

$$\begin{cases} Young = \int_0^{25} 1/x + \int_{25}^{100} [1 + (\frac{x-25}{5})^2]^{-1} / x \\ Old = \int_{50}^{100} [1 + (\frac{x-50}{5})^{-2}]^{-1} / x \end{cases}$$

Continuous



Fuzzy Operations

Some applicable ***fuzzy operations*** are as follows:

- ***Fuzzy Complement***
- ***Fuzzy Union (s-norm)***
- ***Fuzzy Intersection (t-norm)***

Fuzzy Complement

Let $c : [\,0,1\,] \to [\,0,1\,]$ be a mapping that transforms the membership function of fuzzy set A into the membership function of the ***complement*** of A, that is

$$c\left[\mu_A(x)\right] = \mu_{\bar{A}}(x)$$

In order for the function c to be qualified as a complement, it should satisfy at least the following two requirements:

- ***Boundary Condition:***

$$c(0) = 1 \;\;\&\;\; c(1) = 0$$

- ***Non-increasing Condition:***

$$\forall a,b \in [0,1] : if \;\; a < b \Rightarrow c(a) \geq c(b)$$

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Fuzzy Complement

Some well-known fuzzy complements are as follows:

- ***Basic Fuzzy Complement:***

$$c\left(\mu_A(x)\right)=\mu_{\bar{A}}(x)=1-\mu_A(x)$$

- ***Sugeno Class Fuzzy Complement:***

$$c_\lambda\left(\mu_A(x)\right)=\frac{1-\mu_A(x)}{1+\lambda\mu_A(x)}\ ;\ \lambda>-1$$

- ***Yager Class Fuzzy Complement:***

$$c_w\left(\mu_A(x)\right)=\left(1-\mu_A(x)^w\right)^{1/w}\ ;\ w>0$$

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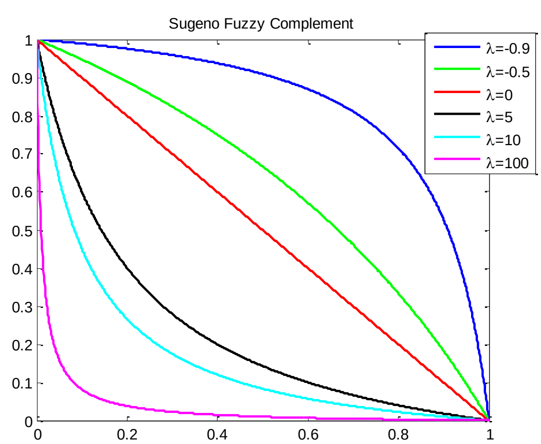
Fuzzy Complement

Example: Sugeno class fuzzy complement for different values of λ:



Fuzzy Union

Let $s:[0,1]\times[0,1]\rightarrow[0,1]$ be a mapping that transforms the membership functions of fuzzy sets A and B into the membership function of the ***union*** of A and B ,that is

$$s\left[\mu_A(x),\mu_B(x)\right]=\mu_{A\cup B}(x)$$

In order for the function s to be qualified as a union, it should satisfy at least the following four requirements:

- ***Boundary Condition:*** $s(1,1)=1,\ s(a,0)=s(0,a)=a$
- ***Commutative Condition:*** $s(a,b)=s(b,a)$
- ***Non-decreasing Condition:*** $a\leq a',\ b\leq b'\Rightarrow s(a,b)\leq s(a',b')$
- ***Associative Condition*** $s(s(a,b),c)=s(a,s(b,c))$

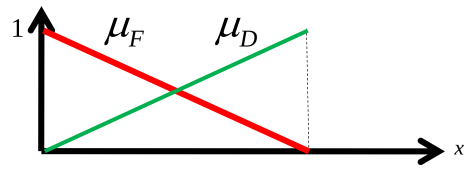
Fuzzy Union

Some well-known *fuzzy unions (s-norms)* are as follows:

- ***Basic Fuzzy Union:*** $\mu_{A\cup B}(x)=\max\{\mu_A(x),\mu_B(x)\}$
- ***Dombi Class Fuzzy Union***
- ***Dubois-Prade Class Fuzzy Union***
- ***Yager Class Fuzzy Union***
- ***Drastic Sum***
- ***Einstein Sum***
- ***Algebraic Sum:*** $\mu_{A\cup B}(x)=\mu_A(x)+\mu_B(x)-\mu_A(x)\mu_B(x)$

Fuzzy Union

Example: Consider fuzzy membership functions of Iranian and foreign cars as follows:



, calculate the fuzzy union $F \cup D$ by employing basic fuzzy union and Yager s-norm for $w = 1$ and $w = 3$.

Solution:

$$Yager\ \ s-norm: \mu_{D \cup F}(x) = s_w\left[\mu_D(x), \mu_F(x)\right] = \min\left[1, \left(\left(\mu_D(x)\right)^w + \left(\mu_F(x)\right)^w\right)^{1/w}\right]$$

Fuzzy Intersection

Let $t: [\,0,1\,] \times [\,0,1\,] \to [\,0,1\,]$ be a mapping that transforms the membership functions of fuzzy sets A and B into the membership function of the ***intersection*** of A and B ,that is

$$t\left[\mu_A(x), \mu_A(x)\right] = \mu_{A \cap B}(x)$$

In order for the function t to be qualified as a intersection, it should satisfy at least the following four requirements:

- ***Boundary Condition:*** $t(0,0) = 0,\ t(a,1) = t(1,a) = a$
- ***Commutative Condition:*** $t(a,b) = t(b,a)$
- ***Non-decreasing Condition:*** $a \le a',\ b \le b' \Rightarrow t(a,b) \le t(a',b')$
- ***Associative Condition*** $t(t(a,b),c) = t(a,t(b,c))$

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Fuzzy Intersection

Some well-known *fuzzy intersections (t-norms)* are as follows:

- ***Basic Fuzzy Intersection:*** $\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$
- ***Dombi Class Fuzzy Intersection***
- ***Dubois-Prade Class Fuzzy Intersection***
- ***Yager Class Fuzzy Intersection***
- ***Drastic Product***
- ***Einstein Product***
- ***Algebraic Product:*** $\mu_{A\cap B}(x) = \mu_A(x)\mu_B(x)$

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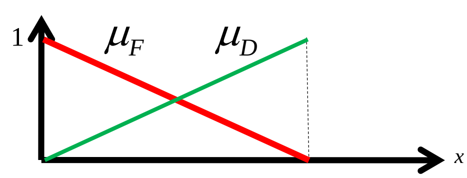
Fuzzy Intersection

Example: Consider fuzzy membership functions of Iranian and foreign cars as follows:



, calculate the fuzzy union $F \cap D$ by employing basic fuzzy intersection and Yager t-norm for $w = 1$ and $w = 3$.

Solution:

$$Yager\ t-norm: \mu_{D\cap F}(x) = t_w\left[\mu_D(x), \mu_F(x)\right] = 1 - \min\left[1, \left(\left(1-\mu_D(x)\right)^w + \left(1-\mu_F(x)\right)^w\right)^{1/w}\right]$$

Fuzzy Operations

Properties: For any s-norm and t-norm, the following inequality holds:

$$\begin{cases} \max(a,b) \le s(a,b) \le s_{ds}(a,b) \\ t_{dp}(a,b) \le t(a,b) \le \min(a,b) \end{cases}$$

where $s_{ds}(a,b)$ and $t_{dp}(a,b)$ are respectively Drastic sum and product

Note that the union and intersection operators cannot cover the interval between min(a, b) and max(a, b). The operators that cover the interval [min(a, b), max(a, b)] are called ***averaging operators***.

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Linguistic Variables

If a variable can take words in natural languages as its values, it is called a ***linguistic variable***, where the words are characterized by fuzzy sets defined in the universe of discourse in which the variable is defined. A ***linguistic variable*** is characterized by:

- **Name**
- **Value**
- **Actual Physical Domain**
- **Semantic Rule**

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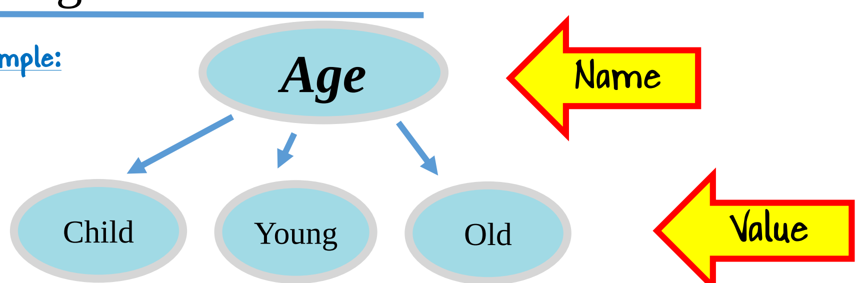
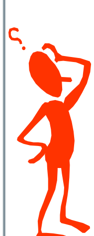
Linguistic Variables

Example:





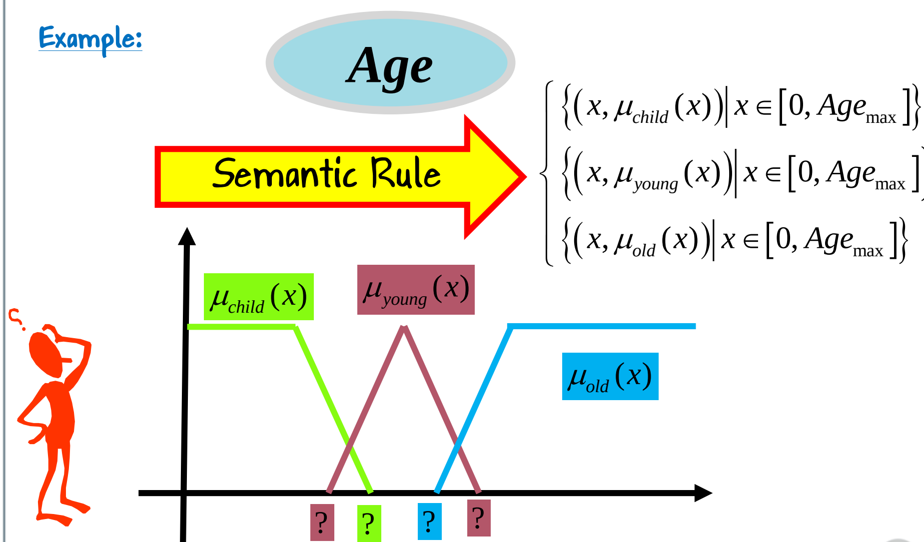
Domain $Age \in [0, Age_{max}]$

Semantic Rule

$$\begin{cases} \{(x, \mu_{child}(x)) \mid x \in [0, Age_{max}]\} \\ \{(x, \mu_{young}(x)) \mid x \in [0, Age_{max}]\} \\ \{(x, \mu_{old}(x)) \mid x \in [0, Age_{max}]\} \end{cases}$$

Linguistic Variables

Example:



$$\begin{cases} \{(x, \mu_{child}(x)) \mid x \in [0, Age_{\max}]\} \\ \{(x, \mu_{young}(x)) \mid x \in [0, Age_{\max}]\} \\ \{(x, \mu_{old}(x)) \mid x \in [0, Age_{\max}]\} \end{cases}$$

Linguistic Variables

The value of a ***linguistic variable*** is a composite term $x = x_1 x_2 \ldots x_n$, that is a concatenation of atomic terms $x_1 x_2 \ldots x_n$. These atomic terms may be classified into three groups:

- ***Primary terms***, which are labels of fuzzy sets such as slow, fast, child
- ***Complement*** "not" and ***connections*** "and" and "or"
- ***Hedges*** such as "very," slightly," "more or less"

Linguistic Variables

Example: Let $U = \{1, 2, 3, 4, 5\}$ and the discrete fuzzy set small be defined as:

$$small = \frac{1}{1} + \frac{0.8}{2} + \frac{0.6}{3} + \frac{0.4}{4} + \frac{0.2}{5}$$

Assume that for any fuzzy set A in U, "very A" and "more or less A" is defined as:

$$\mu_{Very\ A}(x) = [\mu_A(x)]^2 \qquad \mu_{more\&less\ A}(x) = [\mu_A(x)]^{1/2}$$

, obtain very small, very very small, more or less small.

Solution:

Fuzzy Relations

A ***fuzzy relation*** is a fuzzy set defined in the Cartesian product of crisp sets $U_1, U_2, \ldots, U_n$. A ***fuzzy relation*** Q in $U_1 \times U_2 \times \ldots \times U_n$ is defined as the fuzzy set:

$$Q = \left\{ \left((u_1, \ldots, u_n), \mu_Q(u_1, \ldots, u_n) \right) \mid u_1 \in U_1, \ldots, u_n \in U_n \right\}$$

in which μ_Q: $U_1 \times U_2 \times \ldots \times U_n \rightarrow [0,1]$

Fuzzy Relations

Example: Let U and V be the set of real numbers. **A** fuzzy relation "*x is approximately equal to y*," denoted by Q, may be defined by the membership function:

$$\mu_Q(u,v) = e^{-(u-v)^2}$$

Similarly, a fuzzy relation "*x is much larger than y*," denoted by P, may be defined by the membership function

$$\mu_P(u,v) = \frac{1}{1+e^{-(u-v)}}$$

Fuzzy Relations

The ***composition of fuzzy relations*** $P(U,V)$ and $Q(V,W)$, denoted by PoQ , is defined as a fuzzy relation in $U \times W$ whose membership function is given by:

$$\mu_{PoQ}(x,z) = \max_{y \in V}\left\{ t\left[\mu_P(x,y), \mu_Q(y,z) \right] \right\}$$

in which $(x,z) \in U \times W$

Note that Because the t-norm can take a variety of formulas, for each t-norm we obtain a particular composition. The two most commonly used compositions in the literature are the so-called *max-min composition* and *max-product composition*.

Fuzzy Relations

Example: Let U, V and W be the set of real numbers. Consider the fuzzy relation Q (approximately equal) and P (much larger) defined as follows:

$$\mu_Q(u,v) = e^{-(u-v)^2} \qquad \mu_P(u,v) = \frac{1}{1+e^{-(u-v)}}$$

, determine the composition PoQ.

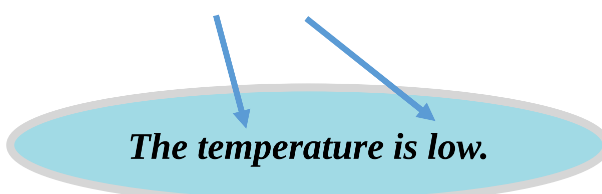
Solution:

Fuzzy Proposition

Let $x \in X$ be a linguistic variable and $A(x)$ be a fuzzy set associated with a linguistic value A. Then the following structure is known as ***atomic fuzzy proposition***:

$$P : x \text{ is } A$$

Example:



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Fuzzy Proposition

The fuzzy proposition can be combined to make more complicate propositions called ***compound fuzzy proposition***, using the connectives "and," "or,“ and "not" which represent fuzzy intersection, fuzzy union, and fuzzy complement, respectively.

And-- --> t-norm
Or ----> s-norm
not ----> complement

Note that fuzzy propositions should be understood as fuzzy relations.

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Fuzzy Proposition

Example: Consider the following fuzzy propositions:

$$\begin{cases} P_1: \quad x \text{ is } A \text{ and } y \text{ is not } B \text{ and } z \text{ is } C \\ P_2: \quad (x_1 \text{ is } A \text{ and } x_2 \text{ is not } B) \text{ or } x_3 \text{ is } C \end{cases}$$

, find their membership functions.

Solution:

Fuzzy Rules

A ***fuzzy IF-THEN rule*** is a conditional statement expressed as:

$$IF \ < Fuzzy\ Proposition > , THEN \ < Fuzzy\ Proposition >$$

Example:

IF *the speed of a car is high,* ***THEN*** *apply less force to the accelerator*

IF *the amount of air is small and it is increased slightly,*
THEN *the surface tension will increase slightly*

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Classical Implication

In classical propositional calculus, the expression "IF p THEN q" is written as:

$$p \rightarrow q$$

in which → is the implication operator. By employing the truth table, it can be easily shown that the p → q is equivalent to:

$$\bar{p} \vee q \quad or \quad (p \wedge q) \vee \bar{p}$$

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Fuzzy Implication

By replacing the p and q in p → q by fuzzy propositions and employing fuzzy operators (s-norm, t-norm and complement), can be easily interpreted fuzzy IF-THEN rules. Assume the fuzzy IF-THEN rule:

$$IF \ < FP1 >, THEN \ < FP2 >$$

where $FP1$ and $FP2$ are respectively fuzzy relations defined in $\mathrm{U} = U_1 \times U_2 \times \ldots \times U_n$ and $\mathrm{V} = V_1 \times V_2 \times \ldots \times V_m$. x and y are linguistic variables (vectors) in U and V, respectively. Accordingly, the following relations stand:

$$\begin{cases} \boxed{\bar{p} \vee q \equiv} & \mu(x,y) = s\left\{c\left(\mu_{FP1}(x)\right), \mu_{FP2}(y)\right\} \\ \boxed{(p \wedge q) \vee \bar{p} \equiv} & \mu(x,y) = s\left\{t\left\{\mu_{FP1}(x), \mu_{FP2}(y)\right\}, c\left(\mu_{FP1}(x)\right)\right\} \end{cases}$$

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Fuzzy Implication

The procedure for assessing the interpretation of the meaning of each fuzzy rule is called ***fuzzy implication***. There are many possible ways to define a fuzzy implication, such as:

- Dienes-Rescher Implication
- Lukasiewicz Implication
- Zadeh Implication
- Gödel Implication

$$p \rightarrow q \equiv \bar{p} \vee q \equiv (p \wedge q) \vee \bar{p}$$

- Mamdani Implication

$$p \rightarrow q \equiv p \wedge q$$

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Fuzzy Implication

- Dienes-Rescher Implication

  $\bar{p} \vee q$: basic fuzzy complement and union

- Lukasiewicz Implication

  $\bar{p} \vee q$: Yager s-norm with w = 1 and basic fuzzy complement

- Zadeh Implication

  $(p \wedge q) \vee \bar{p}$: basic fuzzy complement, union, and intersection

- Gödel Implication

$$\mu(x,y)\begin{cases} 1 & \text{if } \mu_{FP1}(x) \le \mu_{FP2}(y) \\ \mu_{FP2}(y) & o.w. \end{cases}$$

- Mamdani Implication

  $p \wedge q$: basic fuzzy intersection or algebraic product

Fuzzy Implication

Note: ***Mamdani implications*** are the most widely used implications in ***fuzzy systems*** and ***fuzzy control***. They are supported by the argument that fuzzy IF-THEN rules are ***local***. However, one may not agree with this argument. For example, one may argue that when we say "IF speed is high, THEN resistance is high," we implicitly indicate that "IF speed is slow, THEN resistance is low." In this sense, fuzzy IF-THEN rules are ***nonlocal***. This kind of debate indicates that when we represent human knowledge in terms of fuzzy IF-THEN rules, different people have different interpretations. Consequently, different implications are needed to cope with the diversity of interpretations. For example, if the human experts think that their rules are ***local***, then the ***Mamdani implications*** should be used; otherwise, the ***global implications*** should be considered.

Fuzzy Implication

Example: Let x_1 be the speed of a car, x_2 be the acceleration, and y be the force applied to the accelerator. Consider the following fuzzy IF-THEN rule:

$$IF\, x_1 \text{ is slow and } x_2 \text{ is small}, THEN\ y \text{ is large}$$

where the domains of x_1, x_2 and y are $U_1 = [0,100], U_2 = [0,30]$, and $V = [0,3]$, respectively. In addition, the membership function for x_1, x_2 and y are assumed as follows:

$$\mu_{slow}(x_1) = \begin{cases} 1 & x_1 \leq 35 \\ \dfrac{55 - x_1}{20} & 35 < x_1 \leq 55 \\ 0 & x_1 > 55 \end{cases}$$

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Fuzzy Implication

$$\mu_{small}(x_2) = \begin{cases} \dfrac{10 - x_2}{10} & 0 \le x_2 \le 10 \\ 0 & x_2 > 0 \end{cases}$$

$$\mu_{large}(y) = \begin{cases} 0 & y \le 1 \\ y - 1 & 1 < y \le 2 \\ 1 & y > 2 \end{cases}$$

By employing algebraic product as t-norm and Dienes-Rescher implication, interpret the fuzzy IF-THEN rule.

Solution:

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Fuzzy Reasoning

The ultimate goal of fuzzy logic is to provide foundations for fuzzy reasoning with imprecise propositions. The fundamental principles are as follows:

- ***Generalized Modus Ponens*** $(p \wedge (p \rightarrow q)) \rightarrow q$
- ***Generalized Modus Tollens*** $(\bar{q} \wedge (p \rightarrow q)) \rightarrow \bar{p}$
- ***Generalized Hypothetical Syllogism***

$$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$$

Note that these fundamental principles are the extension of the commonly used classical inference rules known as *tautology*.

Fuzzy Reasoning

Example: Consider the *modus ponens* classical inference rules:

$$\begin{aligned} Premise\,1 &: x \text{ is } A \\ Premise\,2 &: IF \;\; x \text{ is } A \;\; THEN \;\; y \text{ is } B \\ Conclusion &: y \text{ is } B \end{aligned}$$

$(p \wedge (p \rightarrow q)) \rightarrow q$

Then, the *generalized modus ponens* is as follows:

$$\begin{aligned} Premise\,1 &: x \text{ is } A' \\ Premise\,2 &: IF \;\; x \text{ is } A \;\; THEN \;\; y \text{ is } B \\ Conclusion &: y \text{ is } B' \end{aligned}$$

The closer the A' to A, the closer the B' to B

Fuzzy Reasoning

Example: The intuitive criteria relating Premise 1 and the Conclusion in ***generalized modus ponens*** can be as follows:

$$\begin{aligned} &Premise\,1 &&: x \text{ is } A' &&\leftarrow \quad x \text{ is more or less } A \\ &Premise\,2 &&: IF\ \ x \text{ is } A \ \ THEN\ \ y \text{ is } B && \\ &Conclusion &&: y \text{ is } B' &&\leftarrow \quad y \text{ is } B \end{aligned}$$

Note that this criteria is intuitive criteria because this is not necessarily true for a particular choice of fuzzy sets; this is what *approximate reasoning* means. Although this criteria is not absolutely correct, it does make some sense. it should be viewed as guidelines (or soft constraints) in designing specific inferences.

Thanks