Intelligent Control

Fundamentals in Control Systems Theory

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Content

- Mathematical Preliminaries
- System Descriptions
- Stability Definitions
- Classical Controllers
- Classical Design Tools

Inner product

Let V be a vector space. An inner product on V is a function $\langle\, , \rangle : V \times V \rightarrow R$ such that for all $v, w, z \in V$ and $s, t \in R$ we have:

- $\langle v, v \rangle \geq 0$, and $\langle v, v \rangle = 0$ if and only if $v = 0$
- $\langle v, w \rangle = \langle w, v \rangle$
- $\langle v, sw + tz \rangle = s\langle v, w \rangle + t\langle v, z \rangle$

A vector space V with an inner product is called an ***inner product space.***

Inner product

Note: The dot product is an inner product on R^n, called the standard inner product on R^n.

$$u = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}^T$$

$$v = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}^T$$

$$\langle u, v \rangle \triangleq \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

Inner product

Note: An extremely important inner product in applied mathematics, physics, and engineering is:

$$\langle f, g \rangle \triangleq \int_a^b f(x)g(x)dx$$

where f(x) and g(x) are continuous real functions on an interval $[a, b]$.

Norm

Let V be a vector space. A norm on V is a function $\| . \| : V \rightarrow R$ such that for all $v, w \in V$ and $s \in R$ we have:

- $\| v \| \geq 0$, and $\| v \| = 0$ if and only if $v = 0$
- $\| sv \| = |s| \| v \|$
- $\| v + w \| \leq \| v \| + \| w \|$

A vector space V with a norm is called a ***normed space.***

Norm

Note: For a vector $u \in R^n$, the standard norm is as follows::

Standard Norm ⇒ $$\|u\|_p \triangleq \left(\sum_{i=1}^{n} |u_i|^p \right)^{1/p}$$

Standard Norm ⇒ $$\|u\|_p \triangleq \left(|u_1|^p + |u_2|^p + \dots + |u_n|^p \right)^{1/p}$$

where $p \geq 1$ and $\| u \|_p$ is called the $p - norm$ $(or\ l_p - norm)$

Norm

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where $p \geq 1$ and $\| u \|_p$ is called the $p - norm$ $(or\ l_p - norm)$

Norm

Note: The important and applicable standard norms are as follows:

- The $2-norm$ is the usual Euclidean length or RMS value:

$$\|u\|_2 \triangleq \left(|u_1|^2 + |u_2|^2 + \dots + |u_n|^2\right)^{1/2}$$

- The $1-norm$

$$\|u\|_1 \triangleq |u_1| + |u_2| + \dots + |u_n|$$

- The $\infty-norm\ (sup-norm)$

$$\|u\|_\infty \triangleq \lim_{p\to\infty}\left(|u_1|^p + |u_2|^p + \dots + |u_n|^p\right)^{1/p} \Rightarrow \|u\|_\infty \triangleq \max_{1\le i\le n}\{|u_i|\}$$

Norm

Note: for a continuous real function f(x) on an interval $[a, b]$, the important standard norms are obtained as follows:

- The $2-norm$ is the usual Euclidean length or RMS value:

$$\|f(t)\|_1 \triangleq \int_a^b |f(t)|\,dt$$

- The $1-norm$

$$\|f(t)\|_2 \triangleq \left(\int_a^b (f(t))^2\,dt\right)^{1/2}$$

- The $\infty-norm\ (sup-norm)$

$$\|f(t)\|_\infty \triangleq \sup_{a\le t\le b}\{|f(t)|\}$$

Norm

Note: Matrix norms are functions $\|.\|: R^{m\times n} \rightarrow R$ that satisfy the same properties as vector norms. For a matrix $A \in R^{m\times n}$, a few examples of matrix norms are:

- The Frobenius norm:

$$\|A\|_F = \sqrt{tr\left(A^T A\right)} = \sqrt{\sum_{i,j} a_{i,j}^2}$$

- The sum-absolute-value norm:

$$\|A\|_{sav} = \sum_{i,j} \left|a_{i,j}\right|$$

- The max-absolute-value norm:

$$\|A\|_{sav} = \max_{i,j} \left|a_{i,j}\right|$$

Norm

Example: Consider $\mathrm{f(t)} = e^{-at}$ $(a > 0)$ on an interval $(0, \infty)$. Calculate the $1 - norm$, $2 - norm$ and $\infty - norm$ of $\mathrm{f(t)}$.

Solution:

Kronecker Product

Let A be a $m \times n$ matrix and let B be a $p \times q$ matrix. Then the ***Kronecker product*** $\otimes$ of A and B is that $mp \times nq$ matrix defined by:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & & a_{1n}B \\ a_{21}B & a_{22}B & & a_{2n}B \\ & & & \\ a_{m1}B & a_{m2}B & & a_{mn}B \end{bmatrix}_{mp\times nq}$$

Note: $A \otimes B \neq B \otimes A$

Kronecker Product

Example: Let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

Calculate $A \otimes B, B \otimes A$.

Solution:

Projection

For any vectors $u, v \in R^n$, with $u \neq 0$, we define the projection of v onto u by:

$$proj_u(v) = \frac{u.v}{\|u\|^2} u$$

Quadratic Forms

A ***quadratic form*** on R^n, with corresponding $n \times n$ symmetric matrix A, is a function Q: $R^n \to R$ defined by:

$$Q(x) = x^T Ax \quad ; \forall x \in R^n$$

Example: Consider $V(x) = x_1^2 - 2x_1x_2 + 4x_1x_3 + 5x_2^2 - 6x_3^2$, find corresponding symmetric matrix.

Quadratic Forms

A ***quadratic form*** Q(x) on R^n is:

- Positive definite if $Q(x) > 0$ for all $x \neq 0$
- Negative definite if $Q(x) < 0$ for all $x \neq 0$
- Indefinite if $Q(x) > 0$ for som x *and* $Q(x) < 0$for som x
- Positive semidefinite if $Q(x) \geq 0$ for all x
- Negative semidefinite if $Q(x) \leq 0$ for all x

Quadratic Forms

A ***principal minor*** of order r of an $n \times n$ matrix $A = [a_{i,j}]$ is the determinant of a matrix obtained by deleting $n - r$ rows and $n - r$ columns such that if the i*th* row (column) is selected, then so is the i*th* column (row).

A principal minor is called a ***leading principal minors*** of order *r* (*1*≤ *r* ≤ *n*) if it consists of the first (”leading”) *r* rows and columns of |A|.

Quadratic Forms

Theorem: Let $Q = x^T Ax$ be a quadratic form of n variables and let

$$|A_1| = a_{11} \; , \; |A_2| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, |A_3| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \ldots, |A_n| = |A|$$

be the leading principal minors of matrix A. Then

- Q(x) ***positive definite*** ⇔ all leading principal minors are positive.
- Q(x) ***negative definite*** ⇔ the leading principal minors of even order are positive and those of odd order are negative..

Quadratic Forms

Theorem: Let $Q = x^T Ax$ be a quadratic form of n variables and let

$$|A_i| = a_{ii} \; , \; |A_2| = \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}, |A_3| = \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix}, \ldots, |A_n| = |A| = 0 \quad (i < j < k)$$

be the principal minors of ***singular*** matrix A. Then

- Q(x) ***positive semidefinite*** ⇔ All the principal minors are positive or zero.
- Q(x) ***negative semidefinite*** ⇔ All the principal minors are of even order are positive or zero and those of odd order are negative or zero.

Quadratic Forms

Example: Let

$$V(x) = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 + 2kx_1x_3 - 2x_2x_3$$

, find k such that V(x) be positive definite.

Solution:

Vector Calculas

The ***differentiation*** of a scalar function $J(x_1, x_2, \dots, x_n)$ with respect to x $= [x_1 \quad x_2 \quad \cdots \quad x_n]^T$ is calculated as follows:

$$\frac{\partial J}{\partial x} = \begin{bmatrix} \partial J / \partial x_1 \\ \partial J / \partial x_2 \\ \vdots \\ \partial J / \partial x_n \end{bmatrix}$$

Vector Calculas

The ***differentiation*** of a vector function $f(x_1, x_2, \dots, x_n) = [f_1 \quad f_2 \quad \cdots \quad f_m]^T$ with respect to $\mathrm{x} = [x_1 \quad x_2 \quad \cdots \quad x_n]^T$ is a $m \times n$ ***Jacobian matrix*** of the following form:

Jacobian Matrix

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \cdots & \partial f_1/\partial x_n \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \cdots & \partial f_2/\partial x_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial f_m/\partial x_1 & \partial f_m/\partial x_2 & \cdots & \partial f_m/\partial x_n \end{bmatrix}_{m \times n}$$

Vector Calculas

The ***differentiation*** of a scalar function $V(x(t))$ with respect to t is calculated by the following ***chain rule***:

$$\frac{d}{dt}\left(V(x(t))\right) = \left(\frac{\partial V}{\partial x}\right)^T \frac{dx}{dt}$$

where $\mathrm{x} = [x_1 \quad x_2 \quad \cdots \quad x_n]^T$ is a vector.

Example:

Vector Calculas

Let A be a matrix and x, y be vectors, then:

- $\frac{\partial(Ax)}{\partial x} = A$
- $\frac{\partial(x^T Ax)}{\partial x} = Ax + A^T x$, for A be a symmetric real matrix: $\frac{\partial(x^T Ax)}{\partial x} = 2Ax$
- $\frac{\partial(x^T Ay)}{\partial x} = Ay$
- $\frac{\partial(x^T Ay)}{\partial y} = Ax$

Comparison Function

- A scalar continuous function $\alpha(r)$, defined for r $\in [0, a)$, belongs to ***class*** $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It belongs to ***class*** $\mathcal{K}_\infty$, if it is defined for all r ≥ 0 and $\alpha(r) \to \infty$, as $r \to \infty$.

- A scalar continuous function $\beta(r, s)$, defined for r $\in [0, a)$, and s $\in [0, \infty)$, belongs to ***class*** $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.

Comparison Function

Example: Let

$$\alpha(r) = tan^{-1}(r) \; ; \gamma(r) = r^2; \;\; \beta(r,s) = \frac{r}{(ksr+1)}$$

, investigate the class of these functions

Solution:

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Linear Time Invariant Systems

A general linear time invariant dynamical system G with input $u(t)$ and output $y(t)$ can be described by the following differential equation:

$$y^{(n)}(t)+a_{n-1}y^{(n-1)}(t)+\ldots+a_2\ddot{y}(t)+a_1\dot{y}(t)+a_0y(t)$$
$$=u^{(m)}(t)+b_{n-1}u^{(m-1)}(t)+\ldots+b_2\ddot{u}(t)+b_1\dot{u}(t)+b_0u(t)$$

Linear Time Invariant Systems

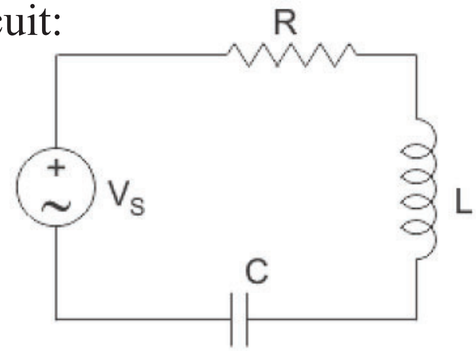
There are two standard descriptions for LTI dynamical systems:

- ***Transfer Function:*** $G(s)=\dfrac{Y(s)}{U(s)}$

- ***State Space:*** $G\begin{cases}\dot{x}(t)=Ax(t)+Bu(t)\\ y(t)=Cx(t)+Du(t)\end{cases}$

Linear Time Invariant Systems

Example: Consider the following RLC circuit:



, find transfer function and state space representations of this LTI system.

Solution:

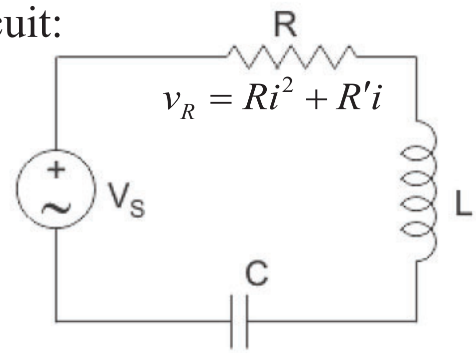
General Dynamical Systems

A general dynamical system G can be described by the following nonlinear time variant state space equations:

$$G\begin{cases}\dot{x}(t) = f\big(x(t), u(t), t\big) \\ y(t) = g\big(x(t), u(t), t\big)\end{cases}$$

General Dynamical Systems

Example: Consider the following RLC circuit:



, find the state space representation of this nonlinear system.

Solution:

Linearization

An ***equilibrium point*** x_{eq} of the nonlinear system $\dot{x} = f\big(x(t), u(t)\big)$ is a point for which $f\big(x_{eq}, u(t) = 0\big) = 0$.

An ***operating point*** x_{op} of the nonlinear system $\dot{x} = f\big(x(t), u(t)\big)$ with a nominal input $u(t) = u_{op}$ is a point for which $f\big(x_{op}, u(t) = u_{op}\big) = 0$.

Note that the operating point and the equilibrium point are usually defined the same

Linearization

Consider a nonlinear system represented as follows:

$$G\begin{cases}\dot{x}(t)=f\left(x(t),u(t),t\right)\\ y(t)=g\left(x(t),u(t),t\right)\end{cases}$$

The corresponding linear model, which defines the system's dynamic behavior about a specific operating point, is:

$$\begin{cases}\Delta\dot{x}(t)=\dfrac{\partial f}{\partial x}\Delta x(t)+\dfrac{\partial f}{\partial u}\Delta u(t)\\ \Delta y(t)=\dfrac{\partial g}{\partial x}\Delta x(t)+\dfrac{\partial g}{\partial u}\Delta u(t)\end{cases}$$

where $\Delta x = x - x_{op}$, $\Delta u = u - u_{op}$ and $\Delta y = y - y_{op}$

Linearization

Example: Consider the following nonlinear system:

$$\begin{cases}\dot{x}_1(t)=x_2(t)\\ \dot{x}_2(t)=\mu x_2(t)+\lambda\sin\left(x_1(t)\right)+u(t)\end{cases}$$

where μ, λ are constant parameters. Find the equilibrium points and its corresponding linear model.

Solution:

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Input-Output Stability

A system is ***BIBO (bounded-input bounded-output) stable*** if every bounded input produces a bounded output.

A SISO system is ***BIBO stable*** if and only if its impulse response g(t) is absolutely integrable in the interval $[0, \infty)$.

Recall that the response of a LTI system is composed of:
response to initial conditions + response to inputs
The concept of Input-Output Stability refers to stability of the response to inputs only, assuming zero initial conditions.

Input-Output Stability

A LTI system with proper rational transfer matrix $G(s) = [G_{ij}(s)]$ is ***BIBO stable*** if and only if every pole of every entry $G_{ij}(s$ of $G(s)$ has negative real part. When the system is represented by state space equations:

$$G\begin{cases}\dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t)\end{cases}$$

, the ***BIBO stability*** will depend on the eigenvalues of the matrix A, since every pole of $G(s)$ is an eigenvalue of A

Note: not every eigenvalue of A is a pole of G(s), since there may be pole-zero cancellations while computing G(s). Thus, a state equation may be BIBO stable even when some eigenvalues of A do not have negative real part.

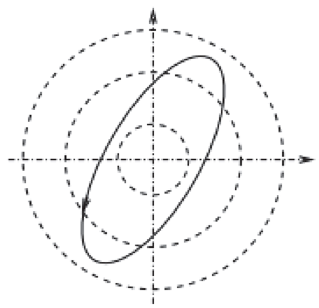
Internal Stability

The system $\dot{x}(t) = Ax(t)$ is ***Lyapunov stable***, or ***marginally stable***, or ***simply stable***, if every finite initial state x_0 excites a bounded response $x(t)$.

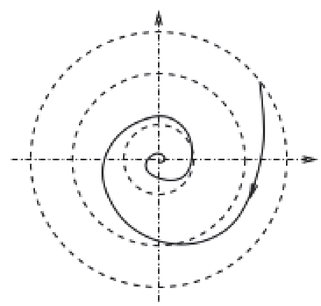
The system $\dot{x}(t) = Ax(t)$ is ***asymptotically stable*** if every finite initial state x_0 excites a bounded response $x(t)$ that approaches 0 as t $\to \infty$

The system $\dot{x}(t) = Ax(t)$ is ***unstable*** if it is not stable

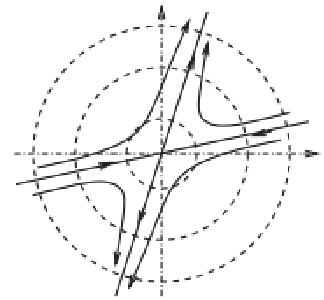
Internal Stability



Lyapunov Stability



Asymptotic Stability



Instability

Internal Stability

The equation $\dot{x}(t) = Ax(t)$ is:

- ***Lyapunov stable*** if and only if all the eigenvalues of A have zero or negative real parts, and those with zero real part are associated with a Jordan block of order 1.
- ***Asymptotically stable*** if and only if all eigenvalues of A have negative real parts.

Lyapunov Theorem

- An equilibrium point $x_{eq} = 0$ is stable ***in the sense of Lyapunov*** if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x(0)\| < \delta$ implies $\|x(t)\| < \varepsilon$ for all $t \geq 0$. Otherwise, the equilibrium point $x_{eq} = 0$ is ***unstable***.
- An equilibrium point $x_{eq} = 0$ is ***asymptotically stable*** if it is stable, and if in addition there exists some $r > 0$ such that $\|x(0)\| < r$ implies that $x(t) \to 0$ as $t \to \infty$.
- An equilibrium point $x_{eq} = 0$ is ***exponentially stable*** if *there exist two strictly positive numbers* α *and* λ *such that*

$$\|x(t)\| \leq \alpha\|x(0)\|e^{-\lambda t} \; ; \; \forall t > 0$$

in some ball around the origin.

Lyapunov Theorem

Note: If asymptotic (or exponential) stability holds for any initial states, the equilibrium point is said to be asymptotically (or exponentially) stable in the large. It is also called ***globally asymptotically (or exponentially) stable***.

Lyapunov Theorem

An equilibrium point $x_{eq} = 0$ is stable ***in the sense of Lyapunov*** if there exists a scalar function $V(x)$ *with continuous first partial derivatives in a region around origin such that*

- $V(x)$ *is positive definite.*
- $\dot{V}(x)$ *is semi-negative definite.*

If, actually, the derivative V(x) is locally negative definite a region around origin , then the stability is ***asymptotic.***

Lyapunov Theorem

Example: Consider the following nonlinear system:

$$\begin{cases} \dot{x}_1 = -x_1 - 2x_2^2 \\ \dot{x}_2 = x_1 x_2 - x_2^3 \end{cases}$$

, evaluate the stability of this system.

Solution:

Lyapunov Theorem

The system $\dot{x}(t) = f(x(t), u(t))$ is ***input-to-state stable*** if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any initial state $x(t_0)$ and any bounded input $u(t)$

$$\|x(t)\| \leq max\left\{\beta(\|x(t_0)\|, t - t_0), \gamma\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\right)\right\}$$

for all $t \geq t_0$

Lyapunov Theorem

Uniformly Ultimately bounded

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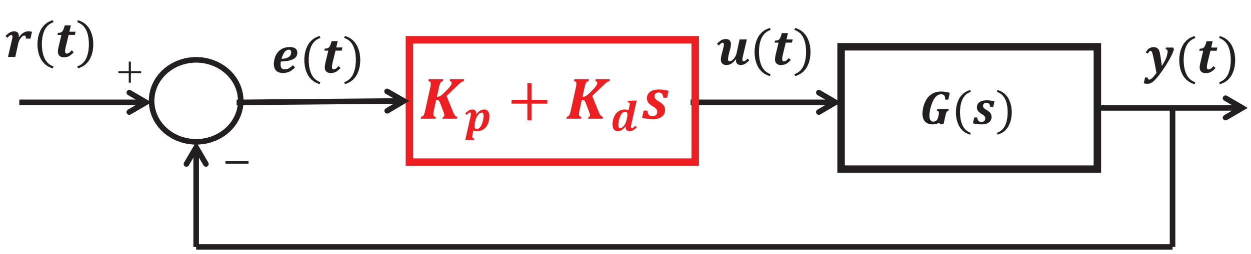
Linear Controllers

- ***PD Controller:*** $K_p + K_d \frac{d(.)}{dt}$

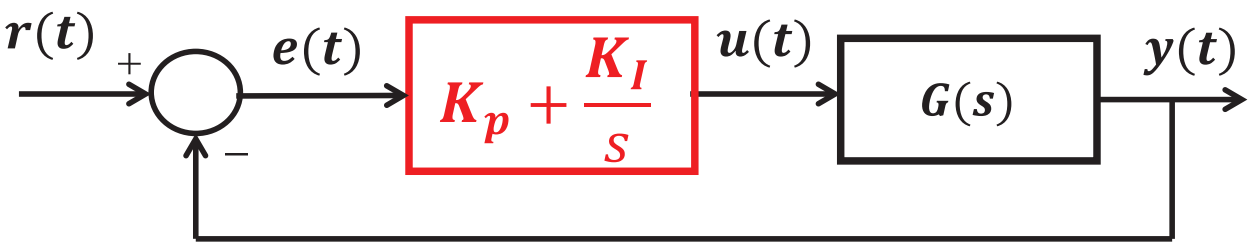


Lead Controller: $K\frac{\alpha s+1}{\beta s+1}$; $for\ \frac{1}{\alpha} < \frac{1}{\beta}$

- *Improve stability*
- *Reduce the overshoot*
- *Decrease the rise time*
- *Increase the system bandwidth*

Linear Controllers

- *PI Controller:* $K_p + K_I \int(.)dt$

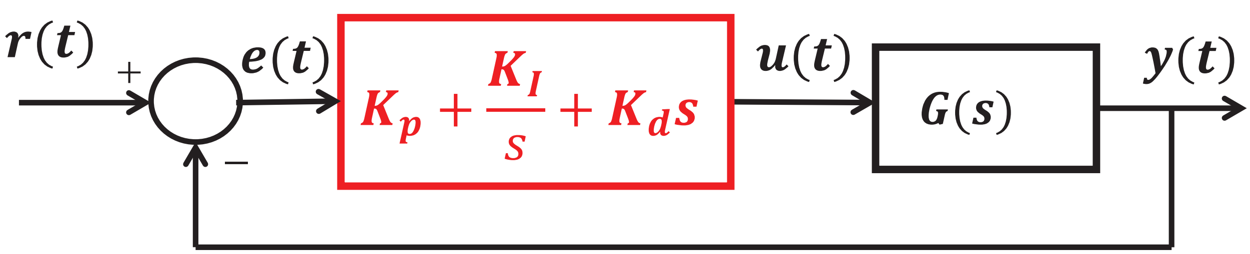


Lag Controller: $K\frac{\lambda s + 1}{\gamma s + 1}$; for $\frac{1}{\lambda} > \frac{1}{\gamma}$

- *Improve the steady-state error*
- *Low-pass filter*
- *Decrease the system bandwidth*

Linear Controllers

- *PID Controller:* $K_p + K_I \int(.)dt + K_d \frac{d(.)}{dt}$

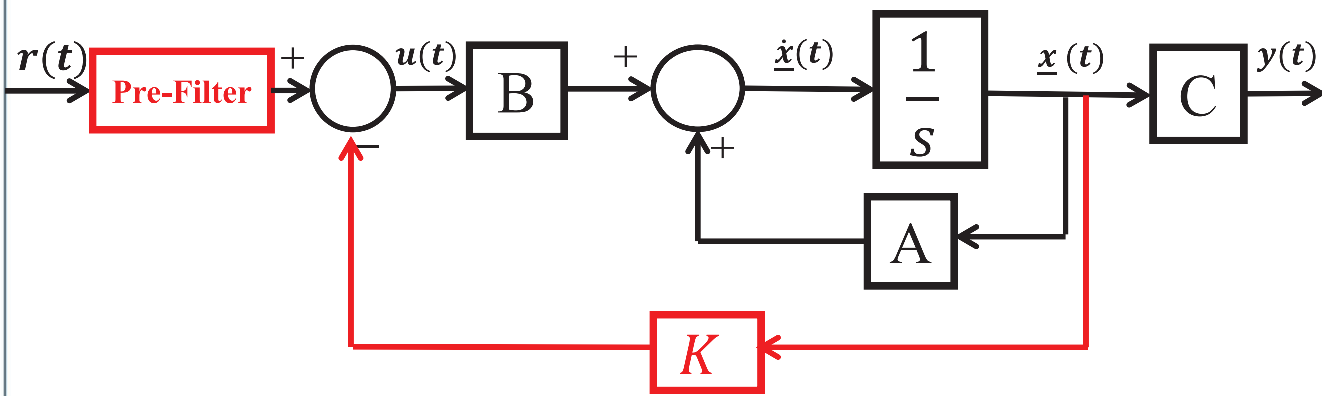


Lead – Lag Controller

- *Improve the steady-state error*
- *Improve the transient response*
- *Band-stop filter*

Linear Controllers

- ***State Feedback:***



$A - BK$: Hurwitz

$$u(t) = -K\underline{x}(t) + \left\{C[-(A-BK)]^{-1}B\right\}^{-1} r(t)$$

Stability **Tracking**

smgrc

Linear Estimator

- ***Observer:***

System

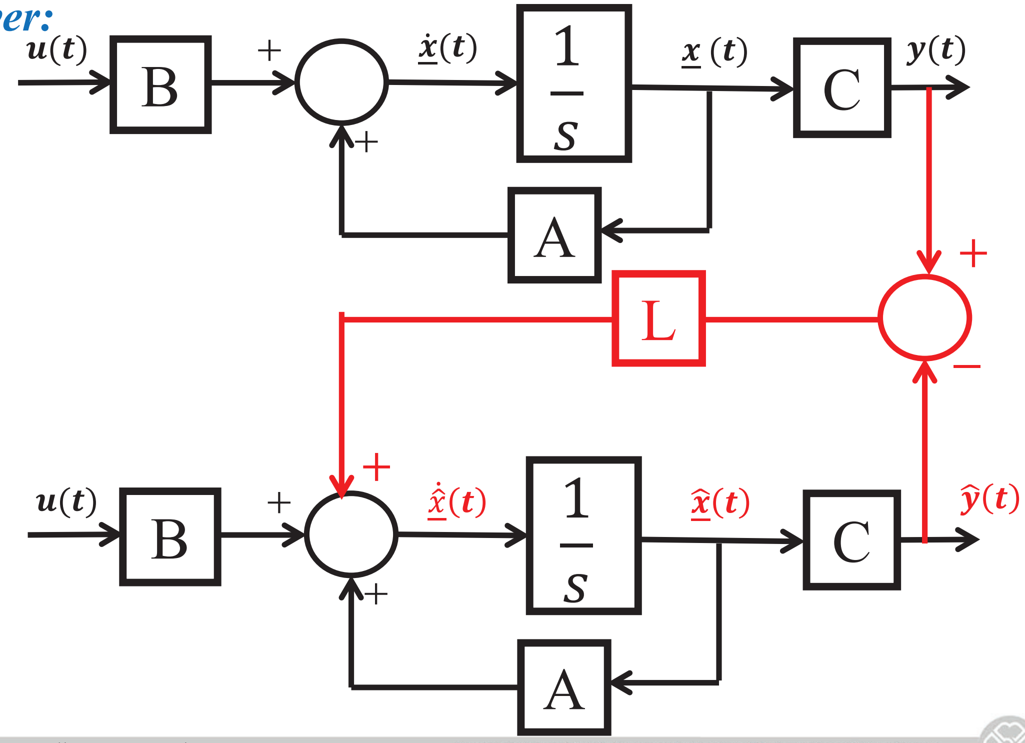
$$\dot{\underline{x}}(t) = A\underline{x}(t) + Bu(t)$$
$$y(t) = C\underline{x}(t)$$

Observer

$$\dot{\hat{\underline{x}}}(t) = A\hat{\underline{x}}(t) + L\left(y(t) - C\hat{\underline{x}}(t)\right) + Bu(t)$$
$$\hat{y}(t) = C\hat{\underline{x}}(t)$$

Linear Estimator

- ***Observer:***



Cost Function Based Linear Controllers

- ***Optimal Controller***
- ***Predictive Controller***
- ***Robust Controller***
- **…..**

Nonlinear Controllers

- *Feedback Linearization*
- *Sliding Mode Controller*
- *Backstepping Controller*
- *Dynamic Surface Controller*
- …..

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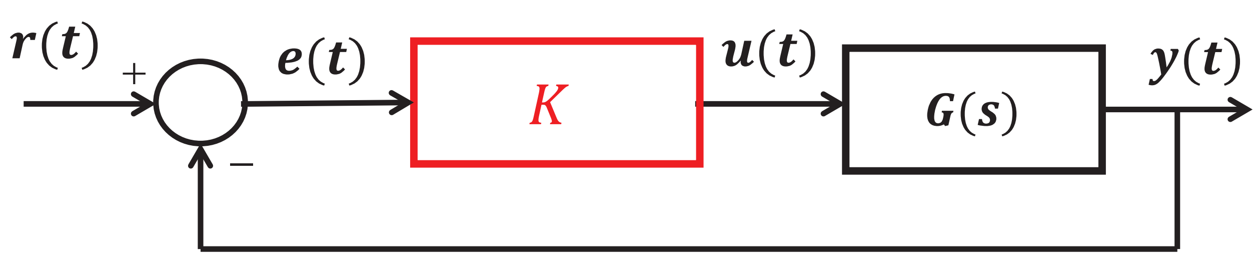
Design Issue

Many issues have to be considered in analysis and design of control systems. The basic requirements are:

- *Stability*
- *Ability to follow reference signals*
- *Reduction of effects of load disturbances*
- *Reduction of effects of measurement noise*
- *Reduction of effects of model uncertainties*

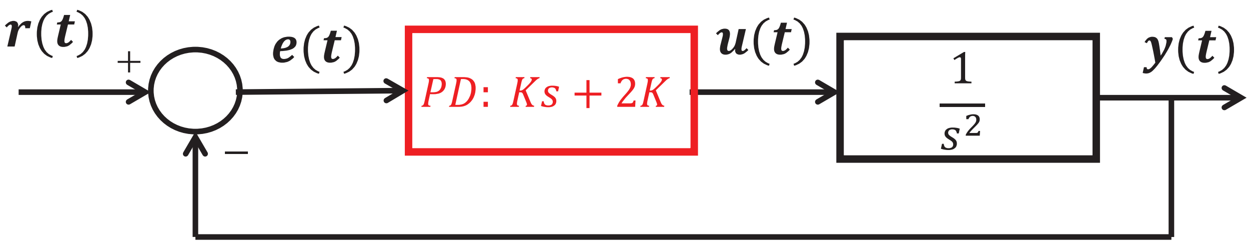
Root Locus Technique

The root locus is used to study the location of the poles of the closed loop transfer function of a given linear system as a function of its parameters, usually a loop gain, given its open loop transfer function.



Root Locus Technique

Example: Consider the following linear system:



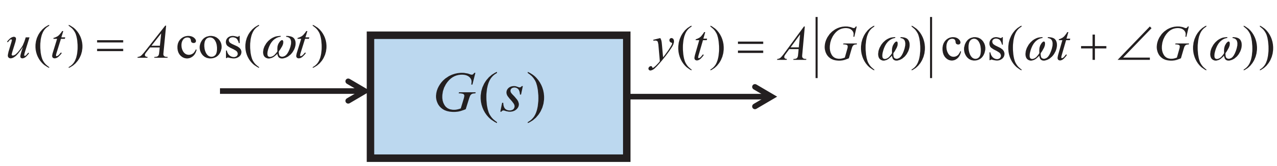
, study the effect of gain variation on the system poles.

Solution:

Frequency Methods

In the ***Bode Plot***, the magnitude in decibels is plotted against the logarithm of the frequency; on a separate plot, the phase in degrees is plotted against the logarithm of the frequency.

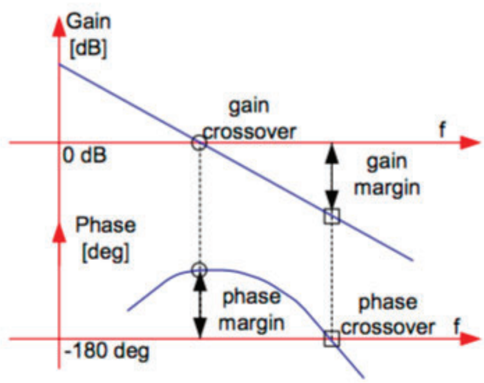
The ***Nyquist plot*** is a plot in the complex plane of Re{G(s)} and Im{G(s)} for $s = j\omega$ as ω goes from zero to infinity.



Frequency Methods

- The ***gain crossover frequency*** ω_g is defined as the frequency at which the total magnitude equals 0 dB.

- The ***phase crossover frequency*** ω_p is defined as the frequency at which the total phase equals -180°.



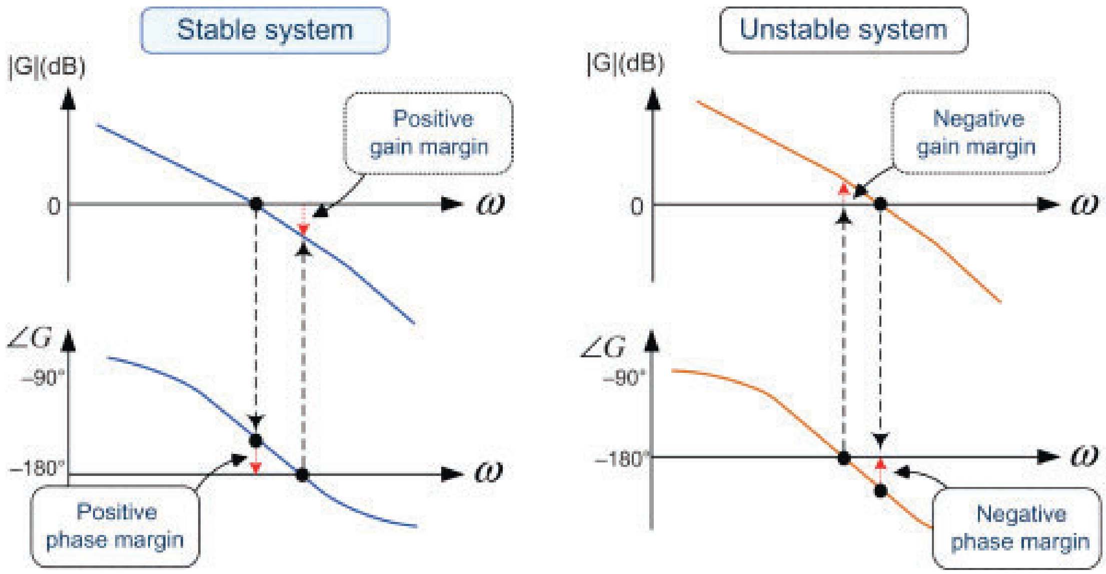
Frequency Methods

- The system ***Gain Margin*** (GM) in dB is the additional gain that makes the system on the edge of instability. GM can be determined by calculating the total magnitude at $\omega = \omega_p$.

- The system ***Phase Margin*** (PM) in degrees is the additional phase that makes the system on the edge of instability. PM can be determined by calculating the total phase at $\omega = \omega_g$.

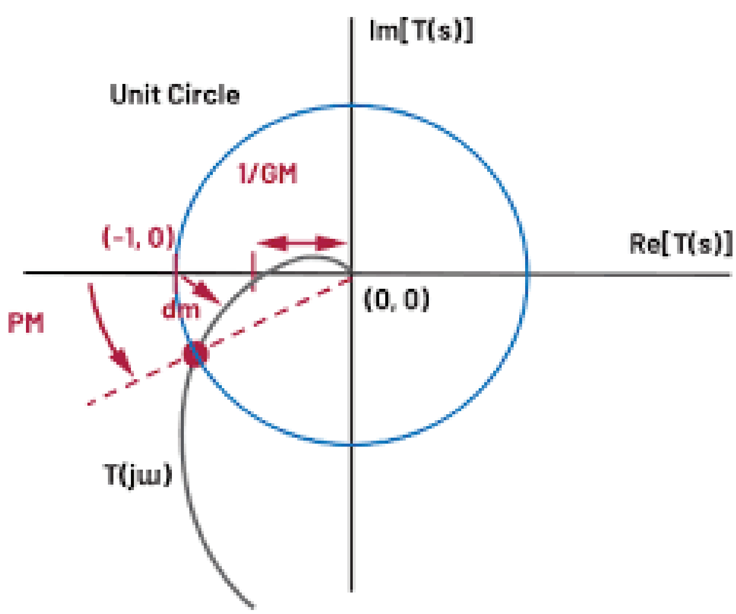
Frequency Methods

Bode Plot



Frequency Methods

Nyquist Plot



Thanks