Robust Control Systems

Robust Stability and Mixed Sensitivity

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Reference

1. S. Skogestadand I. Postlethwaite, **Multivariable Feedback Control; Analysis and Design**, Second Edition, Wiley, 2005.

2. M. Fujita, **Lecture Notes on Feedback Control Systems**, Tokyo Institute of Technology, 2019.

3. R. Smith, **Lecture Notes on Control Systems**, ETH Zurich, 2020.

Robust Stability (RS) Analysis

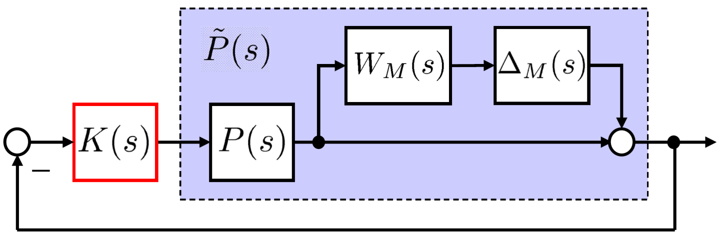
$\tilde{P}(s) = (I + \Delta_M(s)W_M(s))P(s) \qquad \|\Delta_M\|_\infty \leq 1$

$\tilde{P}(s) \in \Pi_0 \quad \Pi_0$: A set of plant models

$P(s)$: Nominal plant model

$W_M(s)$: Uncertainty Weight

$K(s)$: Controller



Robust Stability (RS) Analysis:
Given a controller K, determine whether the system remains stable for all plants in the uncertain set.

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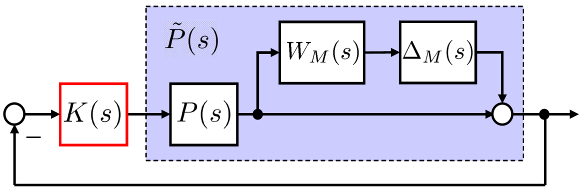
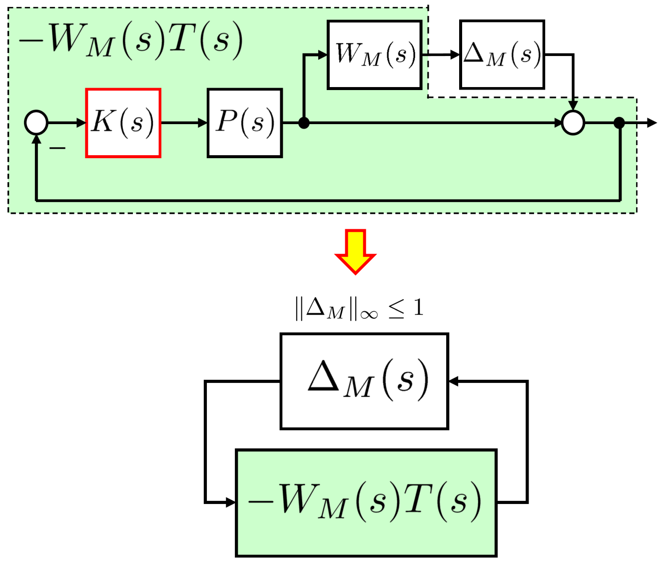
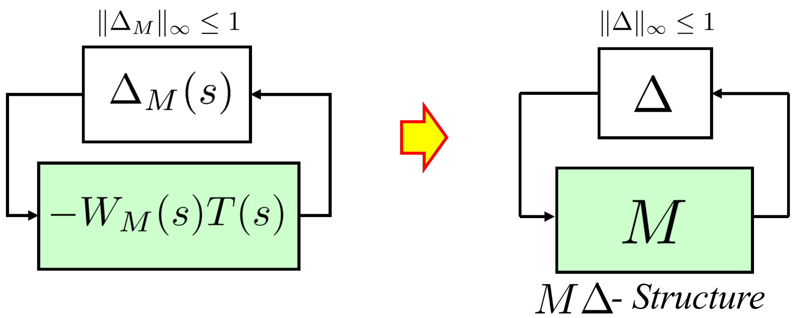




Robust Stability (RS) Test:
Given a controller K, $\|W_M(s)T(s)\|_\infty < 1$
(Small gain theorem)

(Ref 1, pp. 276, 299)

Small Gain Theorem



The closed-loop system is internally stable if $M\Delta$ is stable and satisfies $\|M\Delta\|_\infty < 1$

(Ref 1, pp. 155, 306)

Continue

- Multiplicative Property of $H\infty$ norm allows (System Gain):

$$\|M\Delta\|_\infty \le \|M\|_\infty \|\Delta\|_\infty, \|\Delta\|_\infty \le 1$$

⇓

$$\|\Delta_M\|_\infty \le 1 \text{ and } \|W_M(s)T(s)\|_\infty < 1$$

- **RS Test for SISO Systems:**

$$|T(j\omega)| < \frac{1}{|w_M(j\omega)|}, \forall \omega$$

Note: for H2 norm: $\|M\Delta\|_2 \not\le \|M\|_2 \|\Delta\|_2$

(Ref 1, pp. 155, 306)

Example: RS Test in SISO Systems

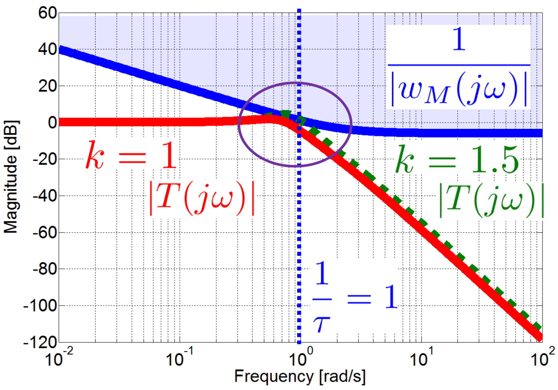
Nominal Plant Model : $P(s) = \dfrac{3}{(s+1)(5s+1)(10s+1)}$

Perturbed Plant Model : $\tilde{P}(s) = e^{-\theta s} P(s), \; 0 \le \theta \le 1$

- Selected Uncertainty Weight: $w_M(s) = \dfrac{2s}{s+2} \quad \begin{pmatrix} 1/\tau = 1 \\ r_\infty = 2 \end{pmatrix}$
- Considered Controller: $K(s) = k \dfrac{(s+0.2)(10s+1)}{s(0.5s+1)}$

$k = 1$ ✔ RS $(k < 1.28)$

$k = 1.5$ ✘ Not RS



(Ref 1, p. 277)

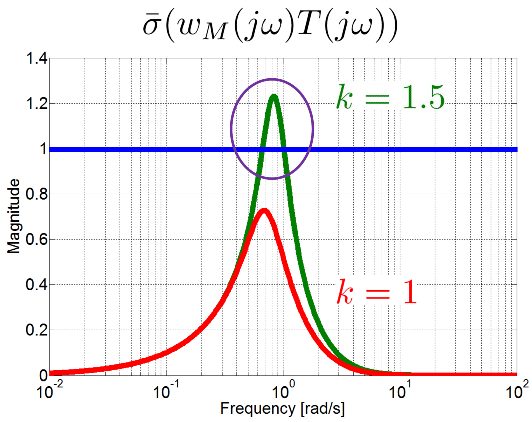
Continue

○ **RS Criteria:**

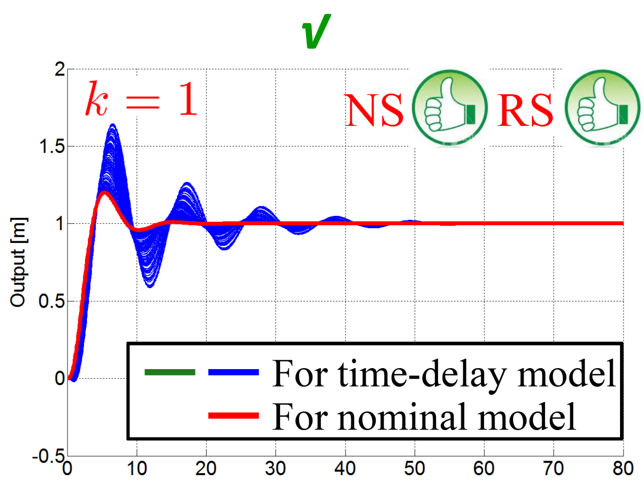
$$\|w_M(s)T(s)\|_\infty < 1$$

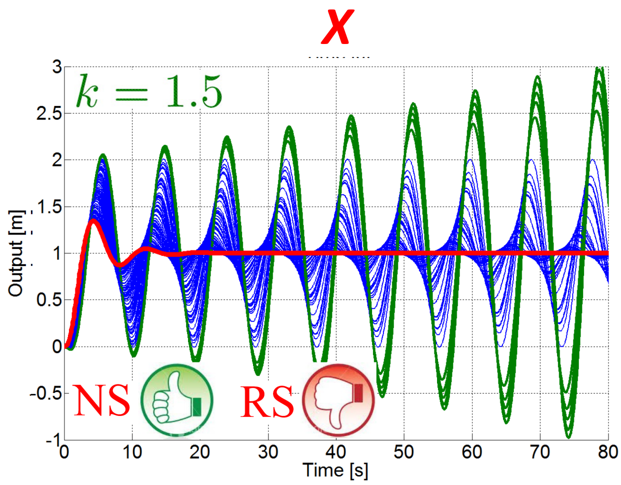
$k = 1 \quad \|w_M(s)T(s)\|_\infty = 0.73$ √

$k = 1.5 \quad \|w_M(s)T(s)\|_\infty = 1.23$ X



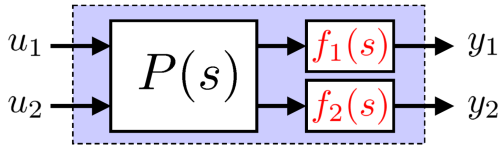
Time (Step) Response Evaluation





Example: RS Test in MIMO Systems

Spinning Satellite



- **Nominal Plant Model :** $P(s) = \begin{bmatrix} \frac{s-100}{s^2+100} & \frac{10s+10}{s^2+100} \\ \frac{-10s-10}{s^2+100} & \frac{s-100}{s^2+100} \end{bmatrix}$

- **Perturbed Plant Model :** $\tilde{P}(s) = \begin{bmatrix} f_1(s) & 0 \\ 0 & f_2(s) \end{bmatrix} P(s)$

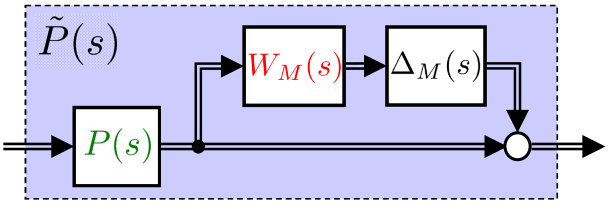
$$f_i(s) = k_i \frac{-\frac{\theta_i}{2}s + 1}{\frac{\theta_i}{2}s + 1}, \quad i = 1,2$$

Gain uncertainty: $0.8 \leq k_i \leq 1.2$

Delay uncertainty: $0 \leq \theta_i \leq 0.02$

Continue

- **Multiplicative (Output) Uncertainty**



$$\Pi_0 = \{\tilde{P}(s) |\ \tilde{P}(s) = (I + \Delta_M(s) W_M(s)) P(s),\ \|\Delta_M\|_\infty \leq 1\}$$

$$W_M(s) = w_M(s) I_2 ,$$
$$w_M(s) = \frac{0.021s + 0.2}{0.0091s + 1}$$

$$(\tau = 0.021,\ r_0 = 0.2,\ r_\infty = 2.3)$$
$$(1/\tau = 48)$$

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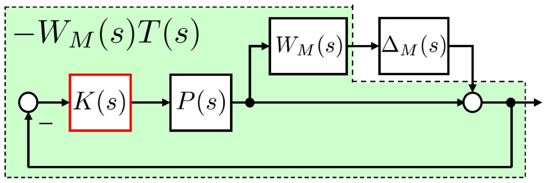
RS: $\|w_M(s)T(s)\|_\infty < 1$



Inverse-based Controller: $K_{\text{inv}}(s) = P^{-1}(s)\begin{bmatrix} \frac{900k}{s(s+30)} & 0 \\ 0 & \frac{900k}{s(s+30)} \end{bmatrix}$

1) $k_1 = 1.0$

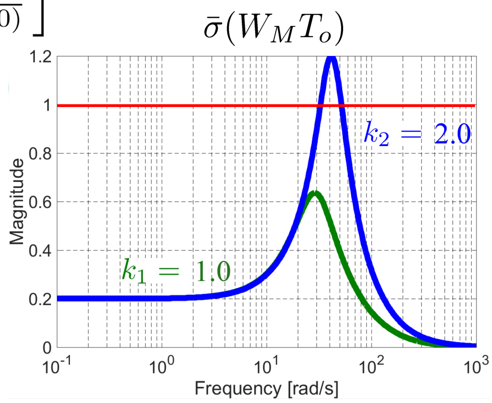
$\|W_M T_o\|_\infty = 0.635$ ✓

2) $k_2 = 2.0$

$\|W_M T_o\|_\infty = 1.985$ ✗

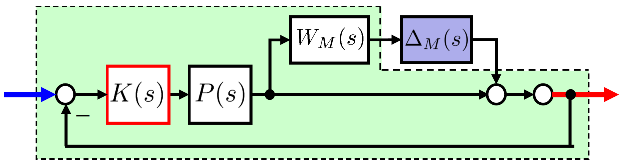
MATLAB Command

```
[SV,w] = sigma(WM*FI.To) ;
hinfTo = normhinf(WM*FI.To)
%hinfTo = max(max(SV))
figure
semilogx(w,SV)
hold on; grid on;
```



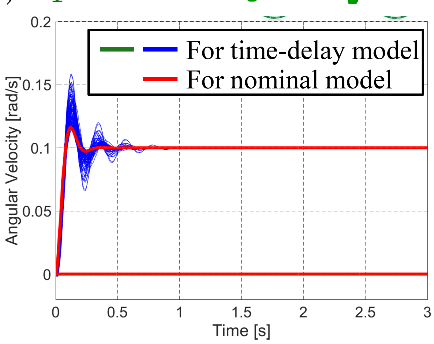
Time (Step) Response Evaluation

$w(t) = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}$

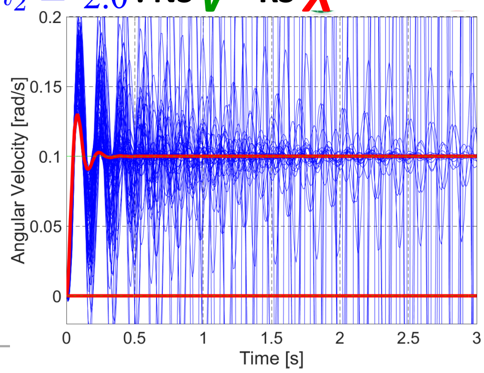


$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$

1) $k_1 = 1.0$: **NS** ✓ **RS** ✓



2) $k_2 = 2.0$: **NS** ✓ **RS** ✗



MATLAB Command

```
time = 0:0.01:3;
step_ref = ones(1,length(time));
ref = [0.1*step_ref';
zeros(1,length(time))];
figure; hold on; grid on;

for i = 1 : 100
    Farray = loopsens(Parray(:,:,i),KI);  [yhi,t] = lsim(Farray.To,ref,time);
    plot(t,yhi(:,1),'b-'); plot(t,yhi(:,2),'g-');
end
FI = loopsens(Pnom,KI);  [yhi,t] = lsim(FI.To,ref,time);
plot(t,yhi,'r-'); plot(time,ref,'g-.');
```

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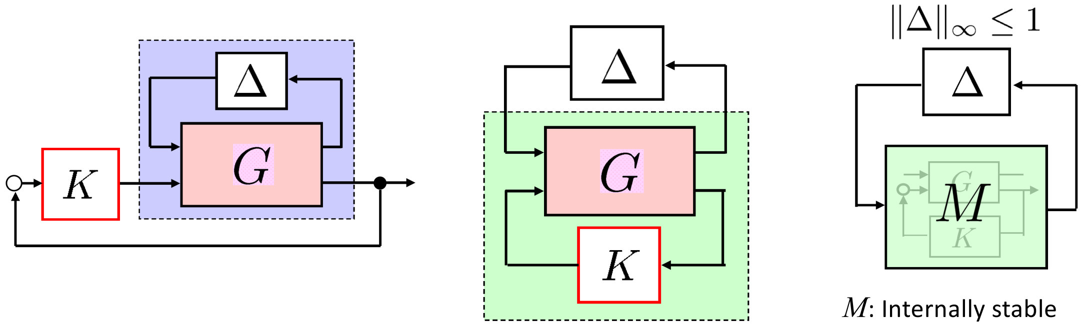
MΔ–Structure and RS Analysis



M: Internally stable
(Nominal stability)

Robust Stability (RS) Test:
Given a controller K, $\|M\|_\infty < 1$
(Small gain theorem)

(Ref 1, pp. 276, 301)

Robust Stability

$$\|M\|_\infty < 1 \, , M = W_1 M_0 W_2$$

Unstr. Uncertainty	Perturbed Model Set $\prod$		M_0
Multiplicative (Output)	$(I + W_2 \Delta W_1) P$	Π_1	$PK(I+PK)^{-1} = T_o$
Multiplicative (Input)	$P(I + W_2 \Delta W_1)$	Π_2	$KP(I+KP)^{-1} = T_i$
Inverse Multip. Output	$(I - W_2 \Delta W_1)^{-1} P$	Π_3	$(I+PK)^{-1} = S_o$
Inverse Multip. Input	$P(I - W_2 \Delta W_1)^{-1}$	Π_4	$(I+KP)^{-1} = S_i$
Additive	$P + W_2 \Delta W_1$	Π_5	$K(I+PK)^{-1} = KS_o$
Inverse Additive	$P(I - W_2 \Delta W_1 P)^{-1}$	Π_6	$(I+PK)^{-1}P = S_o P$

Input Comp. Sens. Func. : $T_i(s) = K(s)P(s)(I + K(s)P(s))^{-1}$

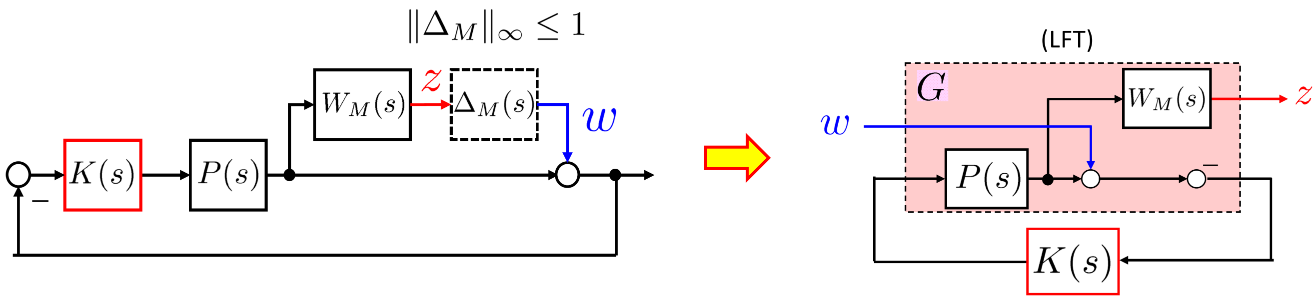
Output Comp. Sens. Func. : $T_o(s) = P(s)K(s)(I + P(s)K(s))^{-1}$

Input Sensitivity Function: $S_i(s) = (I + K(s)P(s))^{-1}$

Output Sensitivity Function: $S_o(s) = (I + P(s)K(s))^{-1}$

(Ref 1, p. 303)

Robust Stabilization (Robust Controller Synthesis)



Robust Stabilization Problem:
Find all stabilizing controllers K, such that $\|W_M(s)T(s)\|_\infty < 1$

Sensitivity Optimization and Robust Stabilization

- **Sensitivity Optimization:**

$$\min_{\text{Feedback } K} \|W_P S\|_\infty = \min_{K} \|W_P(I+PK)^{-1}\|_\infty$$

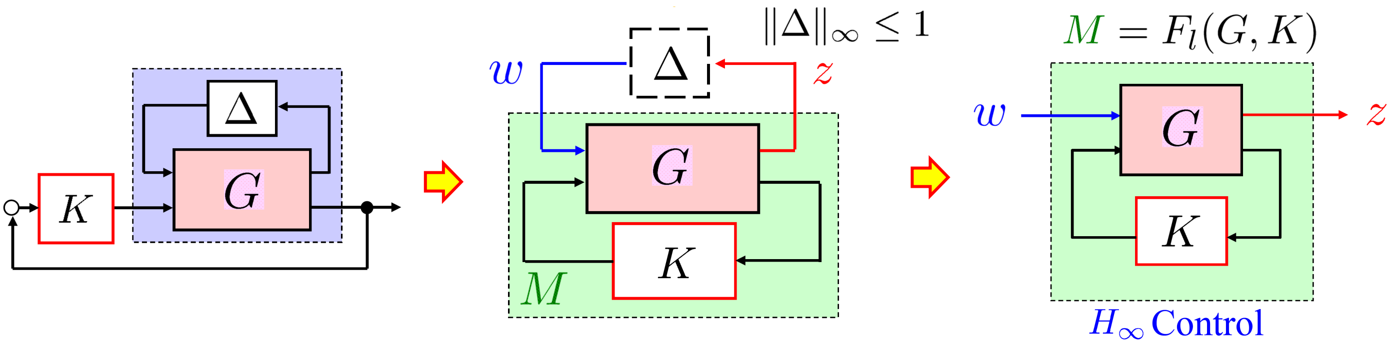
$$\|W_P S\|_\infty < \gamma$$

- **Robust Stabilization:**

$$\min_{\text{Feedback } K} \|W_M T\|_\infty = \min_{K} \|W_M PK(I+PK)^{-1}\|_\infty$$

$$\|W_M T\|_\infty = \gamma^* < 1$$

Robust Stabilization

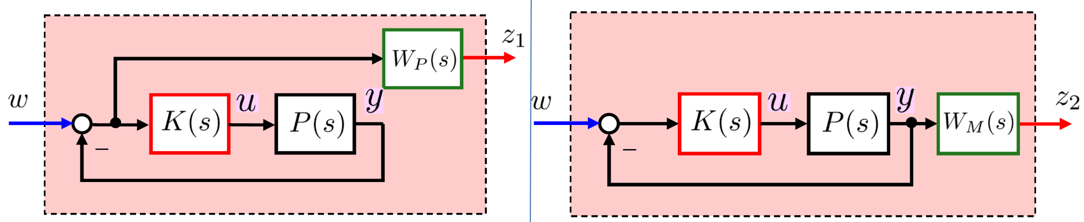


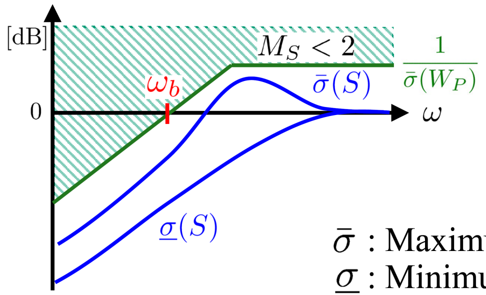
Robust Stabilization Problem:
Find all stabilizing controllers K, such that $\|F_l(G,K)\|_\infty < 1$

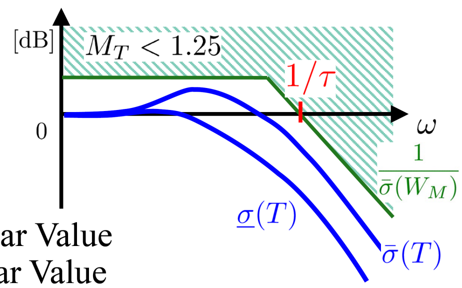
Mixed Sensitivity

Nominal Performance (NP) — Duality and Complementary — **Robust Stability (RS)**

$$\|W_P S\|_\infty < 1 \qquad \|W_M T\|_\infty < 1$$

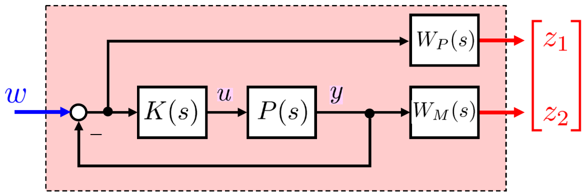




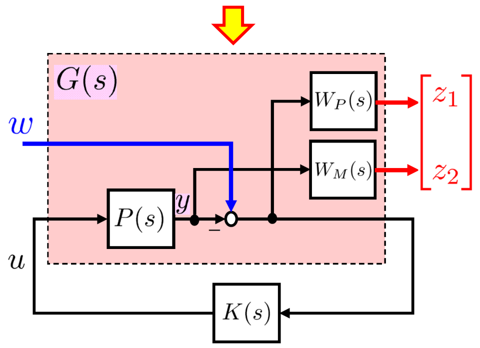


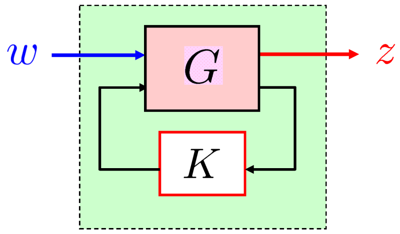
$\bar{\sigma}$: Maximum Singular Value
$\underline{\sigma}$: Minimum Singular Value

Mixed Sensitivity: Stacked Requirements



$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} W_P(s)S(s) \\ -W_M(s)T(s) \end{bmatrix} w$$



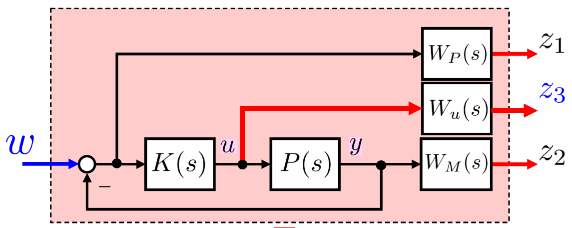


H_∞ Control

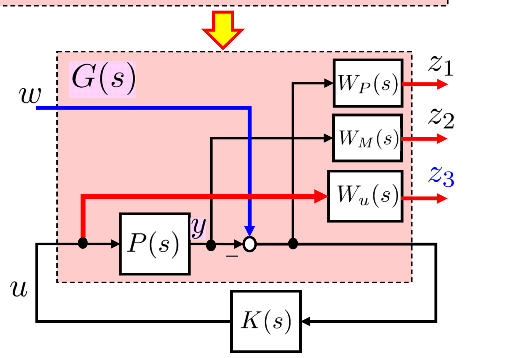
- **Mixed Sensitivity Problem:** Find controllers K, such that: $\left\| \begin{matrix} W_P(s)S(s) \\ W_M(s)T(s) \end{matrix} \right\|_\infty < 1$

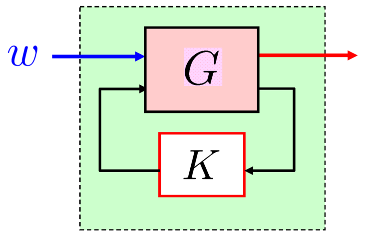
(Ref 1, pp. 62, 282)

$S/T/KS$ Mixed Sensitivity



$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} W_P(s)S(s) \\ W_M(s)T(s) \\ W_u(s)K(s)S(s) \end{bmatrix} w$$





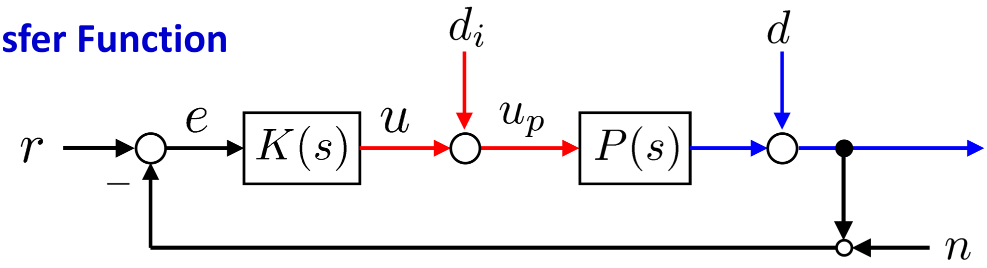
H_∞ Control

- $S/T/KS$ **Mixed Sensitivity Problem:** Find controller K, such that: $\left\| \begin{matrix} W_P S \\ W_M T \\ W_u K S \end{matrix} \right\|_\infty < 1$

(Ref 1, p. 62)

Multivariable Loop Shaping

Loop Transfer Function



Loop Transfer Function at the input to the plant	Loop Transfer Function at the output to the plant
$L_i(s) = K(s)P(s)$	$L_o(s) = P(s)K(s)$

Input Sensitivity Function:

$$S_i(s) = (I + L_i(s))^{-1}$$

Output Sensitivity Function:

$$S_o(s) = (I + L_o(s))^{-1}$$

Input Comp. Sens. Function:

$$T_i(s) = L_i(s)(I + L_i(s))^{-1}$$

Output Comp. Sens. Function:

$$T_o(s) = L_o(s)(I + L_o(s))^{-1}$$

$$L \equiv L_o, \;\; S \equiv S_o, \;\; T \equiv T_o$$

(Ref 1, pp. 341-344)

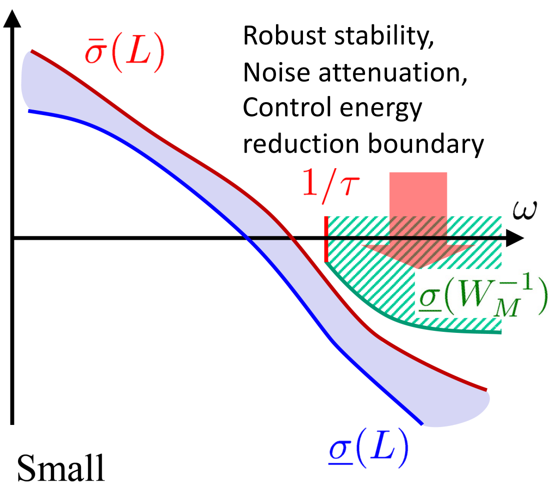
Loop Shaping: For High Frequency

$$T = L(I + L)^{-1}$$

$$\text{If } \bar{\sigma}(L) \ll 1 \, , \;\; \bar{\sigma}(L) \approx \bar{\sigma}(T)$$

$$\text{(RS)} \;\; \bar{\sigma}(T) < \frac{1}{\bar{\sigma}(W_M)} = \underline{\sigma}(W_M^{-1})$$

$$\bar{\sigma}(L) < \underline{\sigma}(W_M^{-1}) \, , \text{ if } \bar{\sigma}(L) \ll 1$$



Open/Closed-loop Objectives ($\bar{\sigma}(L) \ll 1$):

- Noise attenuation: $\bar{\sigma}(T)$, $\bar{\sigma}(L)$ Small
- Input usage (control energy) reduction: $\bar{\sigma}(KS)$, $\bar{\sigma}(K)$ Small
- RS to an additive perturbation: $\bar{\sigma}(KS)$, $\bar{\sigma}(K)$ Small
- RS to a multiplicative output perturbation: $\bar{\sigma}(T)$, $\bar{\sigma}(L)$ Small

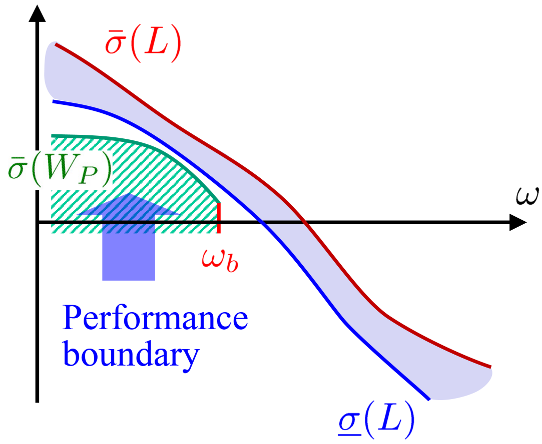
Loop Shaping: For Low Frequency

$$\underline{\sigma}(L) - 1 \le \frac{1}{\bar{\sigma}(S)} \le \underline{\sigma}(L) + 1$$

If $\underline{\sigma}(L) \gg 1$, $\underline{\sigma}(L) \approx \frac{1}{\bar{\sigma}(S)}$

(NP) $\bar{\sigma}(S) < \frac{1}{\bar{\sigma}(W_P)}$

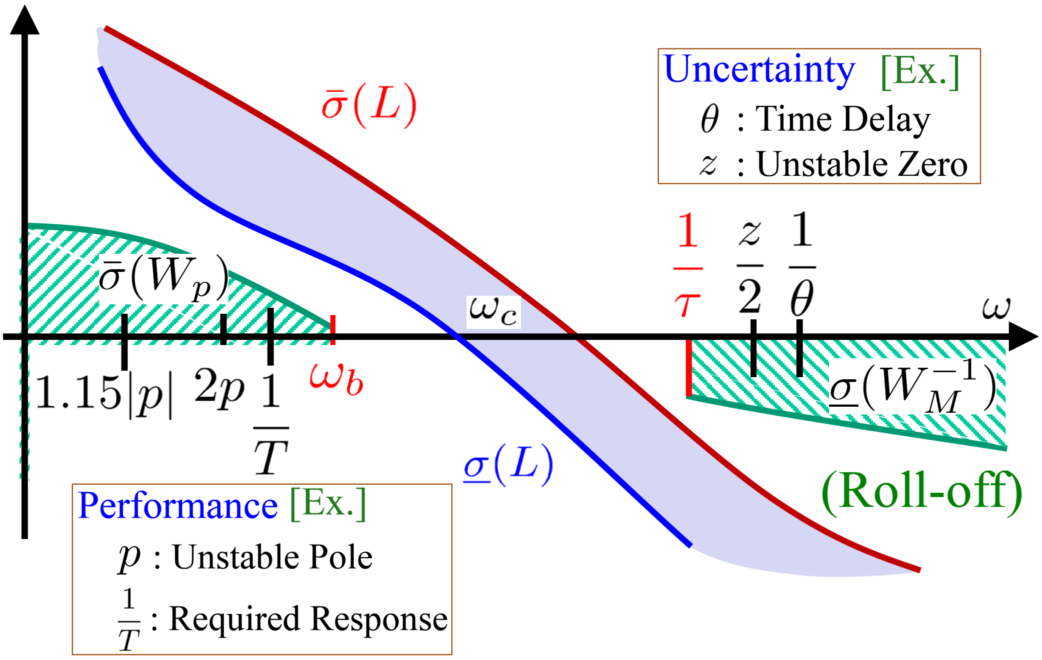
$\underline{\sigma}(L) > \bar{\sigma}(W_P)$, if $\underline{\sigma}(L) \gg 1$



Open/Closed-loop Objectives ($\underline{\sigma}(L) \gg 1$):

- Disturbance rejection: $\bar{\sigma}(S)$ Small , $\underline{\sigma}(L)$ Large
- Reference Tracking: $\bar{\sigma}(T) \approx \underline{\sigma}(T) \approx 1$, $\underline{\sigma}(L)$ Large

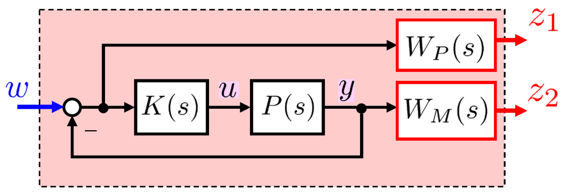
MIMO Loop Shaping



J. C. Doyle and G. Stein, "Multivariable Feedback Design: Concepts for a Classical/ModernSynthesis", IEEE TACSpecial Issue on Linear Multivariable Control Systems, 26-1, 1981

Example: Spinning Satellite

Nominal Model: $P(s) = \begin{bmatrix} \frac{s-100}{s^2+100} & \frac{10s+10}{s^2+100} \\ \frac{-10s-10}{s^2+100} & \frac{s-100}{s^2+100} \end{bmatrix}$



- **Performance and Uncertainty Weights**

$W_P(s) = w_p(s) I_2,\quad w_p(s) = \frac{0.5s + 11.5}{s + 0.115}$

$W_M(s) = w_M(s) I_2,\quad w_M(s) = \frac{0.021s + 0.2}{0.0091s + 1}$

$\omega_b = 11.5,\ M_s = 2,\ A = 0.01 \quad (1/\tau = 48,\ r_0 = 0.2,\ r_\infty = 2.3)$

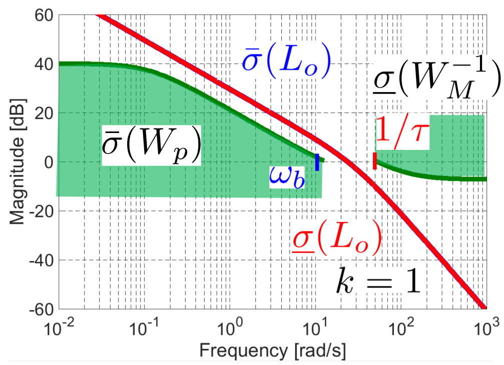
- **Inverse-based Controller**

$K_{\text{inv}}(s) = P^{-1}(s) \begin{bmatrix} \frac{900k}{s(s+30)} & 0 \\ 0 & \frac{900k}{s(s+30)} \end{bmatrix}$

$0.40 \le k \le 1.64$ **NS ✓ RS ✓**

MATLAB Command

sigma(Fl.Lo,WP,inv(WM))

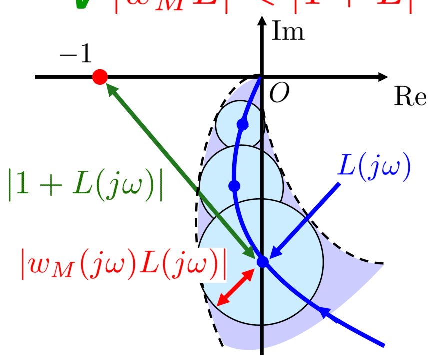


Robust Stability in SISO Systems

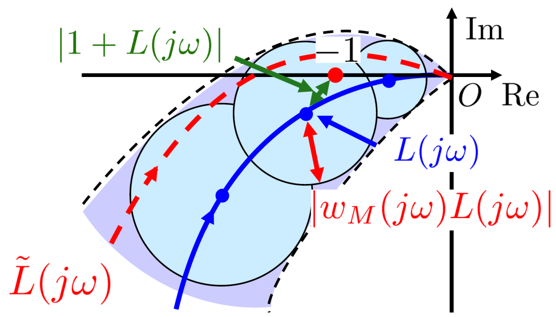
$$\|w_M T\|_\infty < 1 \quad \Leftrightarrow \quad |w_M L| < |1 + L|,\ \forall \omega \qquad T = \frac{L}{1+L}$$

Nyquist Plot:

✓ $|w_M L| < |1 + L|\ \forall \omega$



✗ $|w_M L| > |1 + L|\ \exists \omega$



$\tilde{L}$ should not encircle the point -1, $\forall \tilde{L}$

$\tilde{L} = \tilde{P}K = L + w_M L \Delta_M \quad \|\Delta_M\|_\infty \le 1$

Mixed Sensitivity: Stacked Requirements

Mixed Sensitivity:

$$\left\| \begin{matrix} W_P(s)S(s) \\ W_M(s)T(s) \end{matrix} \right\|_\infty < 1 \quad \begin{cases} \text{NP: } \|W_P S\|_\infty < 1 \\ \text{RS: } \|W_M T\|_\infty < 1 \end{cases}$$

$$\left\| \begin{matrix} w_P S \\ w_M T \end{matrix} \right\|_\infty = \max_\omega \bar{\sigma}\left(\begin{bmatrix} w_P(j\omega)S(j\omega) \\ w_M(j\omega)T(j\omega) \end{bmatrix} \right) < 1$$

$$\max\{\bar{\sigma}(W_P S), \bar{\sigma}(W_M T)\} \le \bar{\sigma}\left(\begin{bmatrix} W_P S \\ W_M T \end{bmatrix} \right) \le \sqrt{2}\max\{\bar{\sigma}(W_P S), \bar{\sigma}(W_M T)\}$$

- **For SISO Systems:**

$$\bar{\sigma}\left(\begin{bmatrix} w_P S \\ w_M T \end{bmatrix} \right) = \sqrt{|w_P S|^2 + |w_M T|^2}$$

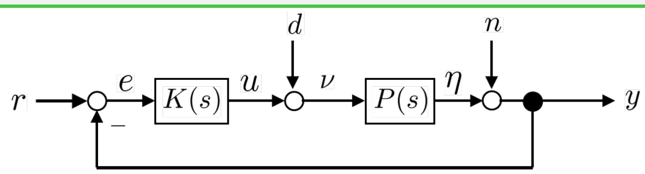
SISO Robust Performance: $|w_P(j\omega)S(j\omega)| + |w_M(j\omega)T(j\omega)| < 1, \ ^\forall\omega$

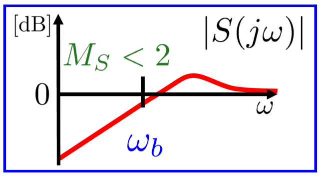
(Ref 1, pp. 282-285)

Loop Shaping

Loop Transfer Function: $L(s) = P(s)K(s)$

$$S = \frac{1}{1+L} \qquad T = \frac{L}{1+L}$$



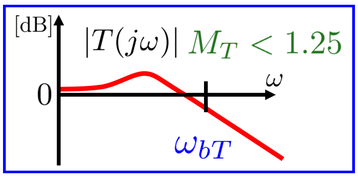


$|L| \gg 1 \rightarrow |S| \ll 1$

large small

$+$

Constraint
$S + T = 1$



$|L| \ll 1 \rightarrow |T| \ll 1$

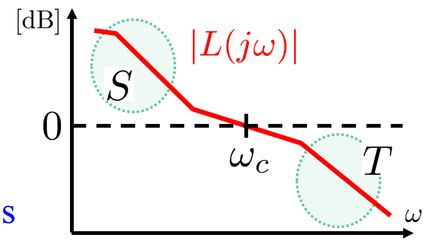
small small

Loop Shaping

Closed-loop S, T

Open Loop L

Stability, Performance, Robustness

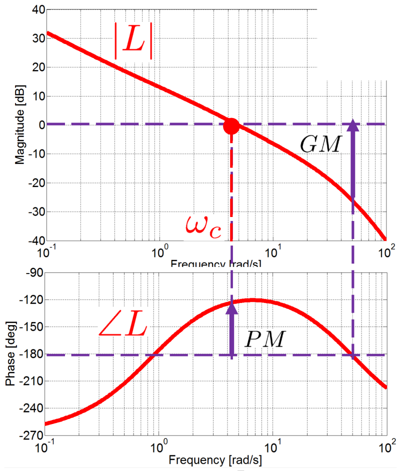


Example

$$L(s) = P(s)K(s)$$

$$P(s) = \frac{4}{(s-1)(0.02s+1)^2}$$

$$K(s) = 1.25\left(1 + \frac{1}{1.25s}\right)$$



- Gain Crossover Frequency: $\omega_c = 4.9$ [rad/s] $|L(j\omega_c)| = 1$
- Stability Margins:

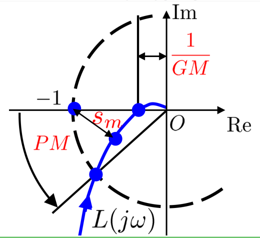
Gain Margin: $GM: \ 2 \sim 5 \ (6 \sim 14 \text{ dB})$

Phase Margin: $PM: \ 30^\circ \sim 60^\circ$

Time Delay Margin: $\theta = PM/\omega_c$

Stability Margin: $s_m = 1/M_S \ : \ 0.5 \sim 0.8$

$GM = 18.7 \quad PM = 59.5^\circ$



(Ref 1, pp. 32, 34)

Continue

- Frequency Domain Performance

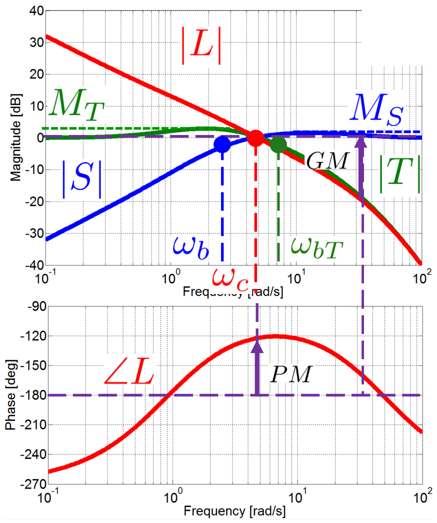
$M_S = 1.19$ $M_T = 1.38$

$M_S < 2$ $M_T < 1.25$

$\omega_b = 2.6$ [rad/s] $\omega_{bT} = 7.8$ [rad/s]

$\omega_c = 4.9$ [rad/s]

$\omega_b < \omega_c < \omega_{bT} \ (PM < 90^\circ)$ $GM = 18.7 \quad PM = 59.5^\circ$



- Maximum Peak Criteria

$$GM \geq \frac{M_S}{M_S - 1}, PM \geq 2\sin^{-1}\left(\frac{1}{2M_S}\right) \text{[rad]}$$

[Ex.] $M_S = 2$ $\rightarrow GM \geq 2, \ PM \geq 29.0^\circ$

$$GM \geq 1 + \frac{1}{M_T}, PM \geq 2\sin^{-1}\left(\frac{1}{2M_T}\right) \text{[rad]}$$

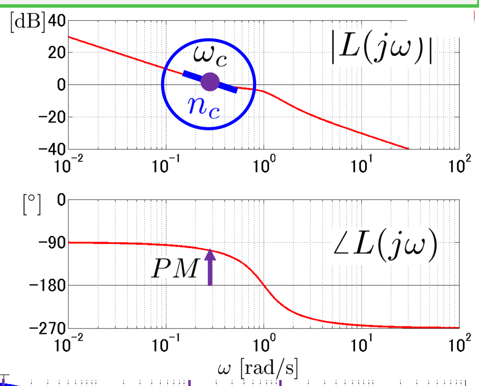
[Ex.] $M_T = 1.25$ $\rightarrow GM \geq 1.8, \ PM \geq 46.0^\circ$

(Ref 1, pp. 34, 36)

Bode Gain-phase Relationship

$$\angle G(j\omega_c) \approx \frac{\pi}{2}\underbrace{\left(\frac{d\ln|G(j\omega)|}{d\ln\omega}\right)_{\omega=\omega_c}}_{n_c}$$

(minimum phase systems)



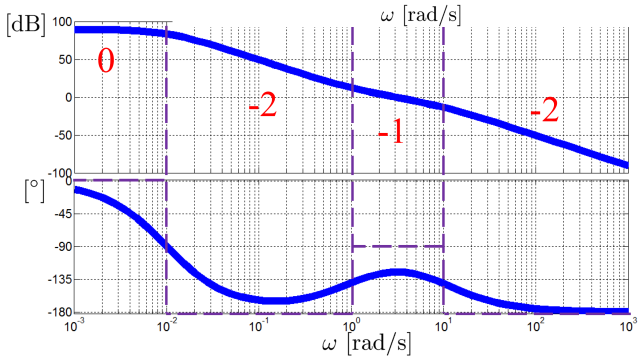
Slope of the Gain Curve at ω_c

$$n_c = -1 \rightarrow \angle G(j\omega_c) = -90^\circ$$
$$n_c = -2 \rightarrow \angle G(j\omega_c) = -180^\circ$$

Sharp Slope: Small Phase Margin

- Example:

$$L(s) = \frac{30(s+1)}{(s+0.01)^2(s+10)}$$

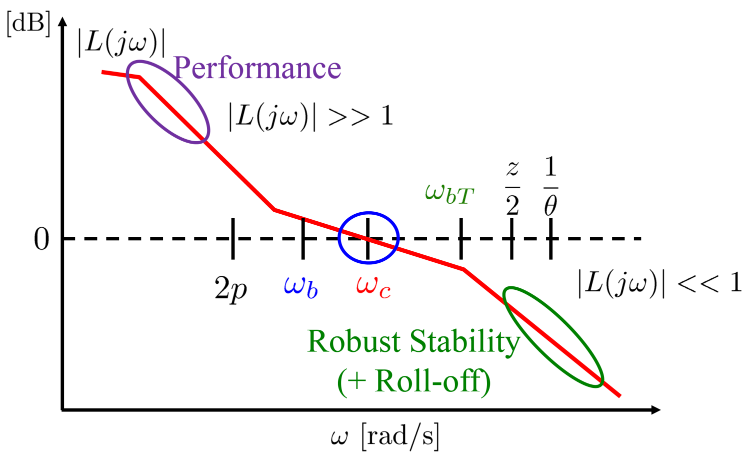


(Ref 1, pp. 18, 20)

SISO Loop Shaping

Loop Shaping Specifications

- Gain Crossover Frequency ω_c
- Shape of $L(j\omega)$
- System Type, Defined as the Number of Pure Integrators in $L(s)$
- Roll-off at Higher Frequencies



(Ref 1, pp. 41, 42, 343)

Thank You!