Robust Control Systems

Structured Singular Value

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Contents

Reference

1. S. Skogestadand I. Postlethwaite, **Multivariable Feedback Control; Analysis and Design**, Second Edition, Wiley, 2005.

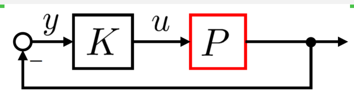
2. M. Fujita, **Lecture Notes on Feedback Control Systems**, Tokyo Institute of Technology, 2019.

3. R. Smith, **Lecture Notes on Control Systems**, ETH Zurich, 2020.

Stability and Performance in MIMO Systems

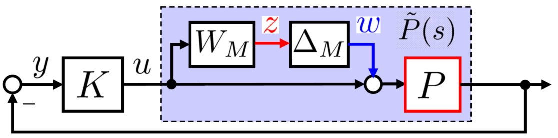
NS: Nominal Stability

Stable S, T, PS, KS



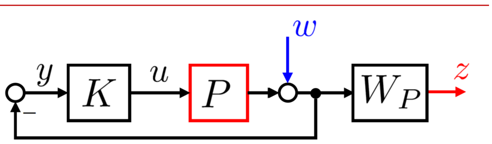
RS: Robust Stability

$\|W_M T\|_\infty < 1$ $\|\Delta_M\|_\infty \leq 1$



NP: Nominal Performance

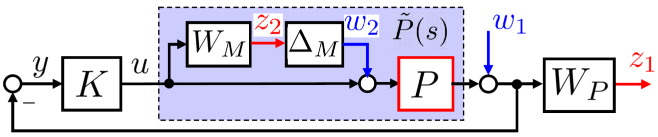
$\|W_P S\|_\infty < 1$



RP: Robust Performance

$\|W_P \tilde{S}\|_\infty < 1 \quad \forall \tilde{P} \in \Pi$

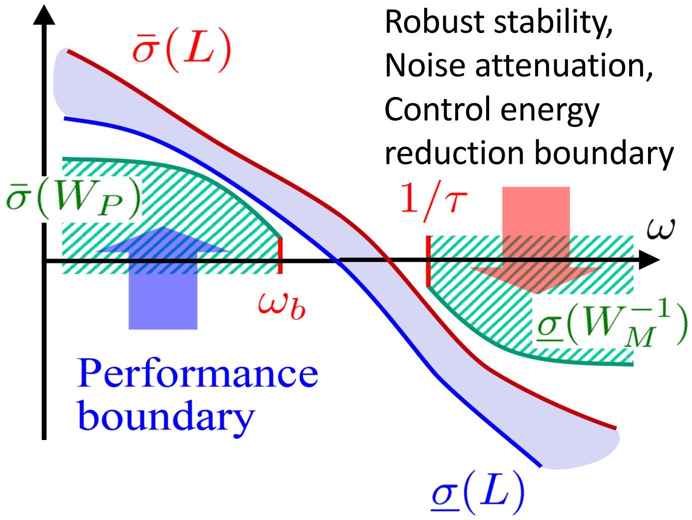
$\tilde{S} = (I + \tilde{P}K)^{-1}$



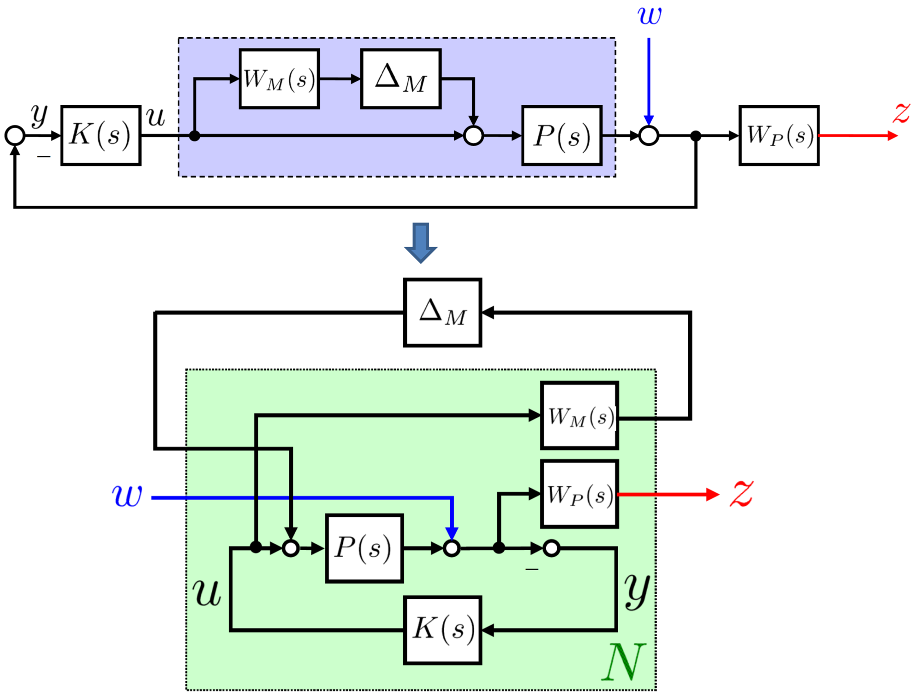
(Ref 1, pp. 300, 316-320)

Structured Singular Value (μ)

Multivariable Loop Shaping Via Singular Values σ → **Structured Singular Value SSV (μ)**



A Framework for Robust Stability/Performance Problems



(Ref 1, p. 298)

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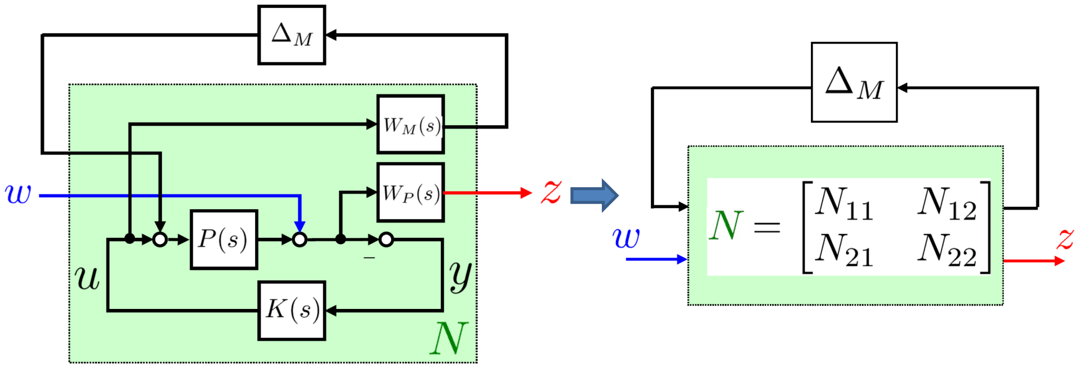
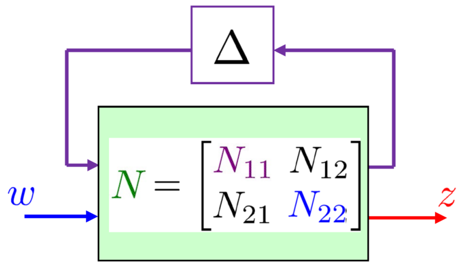


$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} -W_M KP(I+KP)^{-1} & -W_M K(I+PK)^{-1} \\ W_P(I+PK)^{-1}P & W_P(I+PK)^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} -W_M T_I & -W_M K S_o \\ W_P S_o P & W_P S_o \end{bmatrix}$$

(Ref 1, p. 298)

Continue

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} -W_M T_I & -W_M K S_o \\ W_P S_o P & W_P S_o \end{bmatrix}$$



- **LFT:** $z = F_u(N,\Delta)w$

  $F_u(N,\Delta) = N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}$

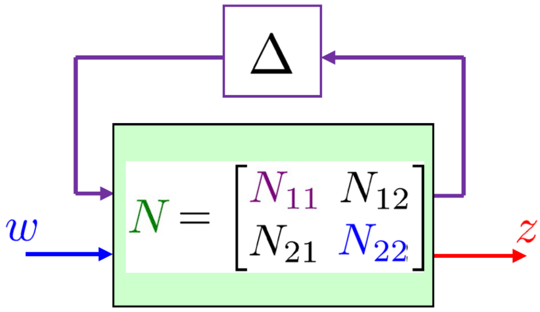
- **Nominal Stability (NS):** *Internally Stable*: S, T, PS, KS

- **Nominal Performance (NP):**

  NS and $\|N_{22}\|_\infty = \|W_P S_o\|_\infty < 1 \quad (\mu(N_{22}(j\omega)) < 1, \ \forall\omega)$

(Ref 1, p. 300)

Continue …



- **RS: Robust Stability**

$$\|N_{11}\|_\infty = \|W_M T_I\|_\infty < 1 \quad (\mu(N_{11}(j\omega)) < 1, \ ^\forall\omega)$$

- **RP: Robust Performance**

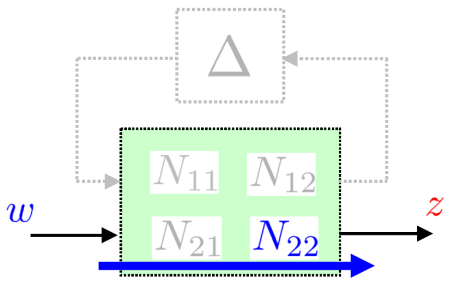
$$\text{RS and } \|F_u(N,\Delta)\|_\infty < 1 \ ^\forall\Delta, \|\Delta\|_\infty \le 1$$

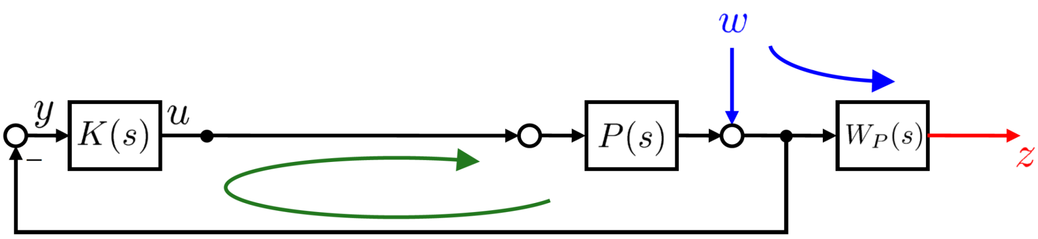
(Ref 1, p. 300)

More Continue

- **Nominal Performance (NP):**

$$\mu(N_{22}(j\omega)) < 1, \ ^\forall\omega$$

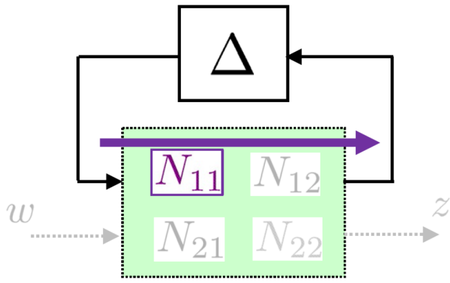


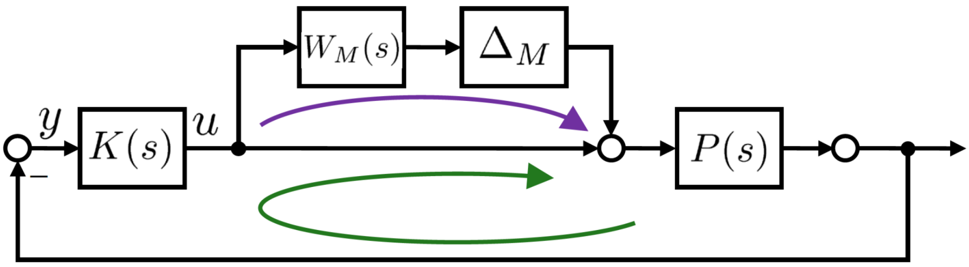


More Continue …

- **Robust Stability (RS):**

$$\mu(N_{11}(j\omega)) < 1, \ ^{\forall}\omega$$

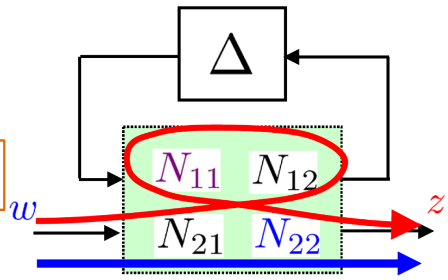




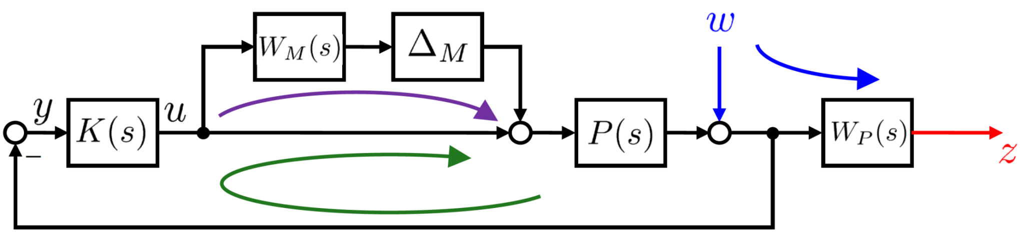
More Continue …

- **Robust Performance (RP):**

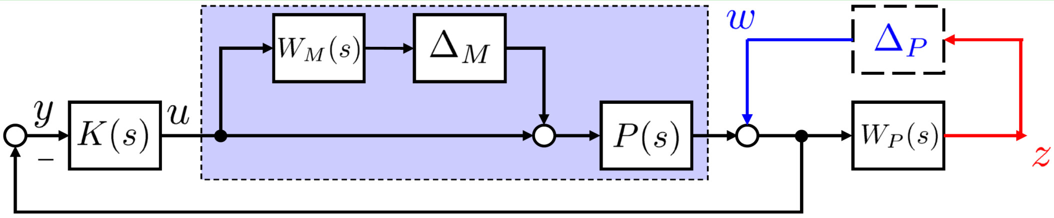
$$\|F_u(N,\Delta)\|_\infty < 1 \ ^{\forall}\Delta, \|\Delta\|_\infty \leq 1$$

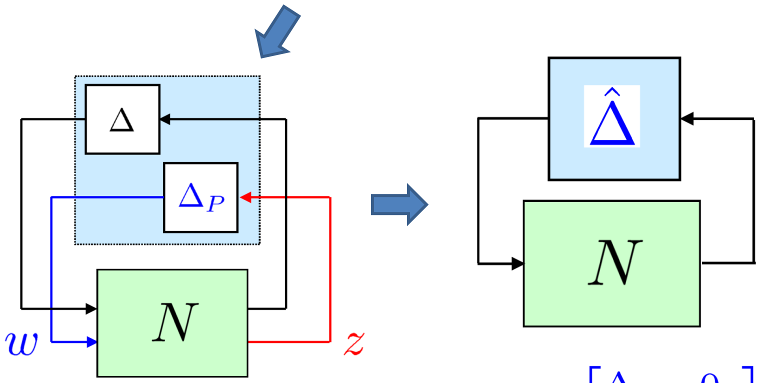


$$\|F_u(N,\Delta)\|_\infty = \|N_{22} + \underline{N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}}\|_\infty$$



Robust Performance and Structured Uncertainties

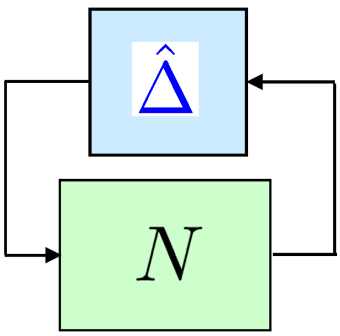




Δ_P **: Fictitious "Performance Block"** $\quad \hat{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_P \end{bmatrix}$

(Ref 1, p. 317)

Robust Performance and Structured Uncertainties



Theorem:

$$\text{RP} \Leftrightarrow \|F_u(N,\Delta)\|_\infty < 1, \ {}^\forall\|\Delta\|_\infty \le 1$$
$$\Leftrightarrow \mu_{\hat{\Delta}}(N(j\omega)) < 1, \ {}^\forall\omega$$

(Ref 1, p. 317)

Structured Singular Value (μ): Definition

For $M \in \mathcal{C}^{n\times n}$, μ_Δ is defined

$$\mu_\Delta(M) := \frac{1}{\min\{\bar{\sigma}(\Delta) | \Delta \in \mathbf{\Delta},\ \det(I - M\Delta) = 0\}}$$

unless no $\Delta \in \mathbf{\Delta}$ makes $I - M\Delta$ singular,
in which case $\mu_\Delta(M) := 0$.

(Ref 1, p. 306)

What is μ?

- **Main Loop Theorem:**

$$\mu_{\hat{\Delta}}(N) < 1 \Leftrightarrow \det(I - N\hat{\Delta}) \neq 0,\ {}^{\forall}\hat{\Delta}, \bar{\sigma}(\hat{\Delta}) \leq 1$$

Proof:

$$\begin{aligned} &\det(I - N\hat{\Delta}) \\ &= \det\left(I - \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_P \end{bmatrix}\right) = \det \begin{bmatrix} I - N_{11}\Delta & -N_{12}\Delta_P \\ -N_{21}\Delta & I - N_{22}\Delta_P \end{bmatrix} \\ &= \det(I - N_{11}\Delta) \cdot \det(I - (N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12})\Delta_P) \\ &= \det(I - N_{11}\Delta) \cdot \det(I - F_u(N, \Delta)\Delta_P) \\ &\neq 0,\ {}^{\forall}\Delta, {}^{\forall}\Delta_P\ \ \|\Delta\|_\infty \leq 1 \end{aligned}$$

(Ref 1, p. 317)

Continue

$$\det(I - N\hat{\Delta}) = \det(I - N_{11}\Delta)\cdot\det(I - F_u(N,\Delta)\Delta_P) \neq 0,\ \forall\Delta, \forall\Delta_P\ \ \|\Delta\|_\infty \leq 1$$

$$\Downarrow$$

$$\mu_{\hat{\Delta}}(N) < 1 \Leftrightarrow \begin{cases} \det(I - N_{11}\Delta) \neq 0,\ \forall\Delta, \bar{\sigma}(\Delta) \leq 1 \\ \det(I - F_u(N,\Delta)\Delta_P) \neq 0,\ \forall\Delta_P, \forall\Delta \end{cases}$$

$$\Leftrightarrow \begin{cases} \mu_\Delta(N_{11}) < 1,\ \forall\omega & \text{Robust Stability} \\ \mu_{\Delta_P}(F_u(N,\Delta)) < 1,\ \forall\omega & \text{Robust Performance} \end{cases}$$

(Ref 1, p. 317)

Structured Singular Value (μ)

$$\mu_\Delta(M) < 1 \iff \text{The } \Delta : \bar{\sigma}(\Delta) > 1 \iff \det(I - N\Delta) \neq 0,\ \forall\Delta,\ \bar{\sigma}(\Delta) \leq 1$$

Stable in Large Δ " Good" $\Leftrightarrow$ Optimal Control Small

Unstable in Small Δ "Bad" $\Leftrightarrow$ Optimal Control Large

- **Example:**

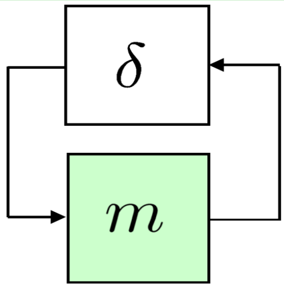
(i) $\mu = 2.0 \quad (> 1)$ $\Leftrightarrow$ $\bar{\sigma}(\Delta) = \frac{1}{2} = 0.5\ (< 1)$ Small

(ii) $\mu = 0.66\cdots (< 1)$ $\Leftrightarrow$ $\bar{\sigma}(\Delta) = \frac{3}{2} = 1.5\ (> 1)$ Large

(Ref 1, p. 306)

Mathematical Properties of μ

○ **μ of a Scalar**



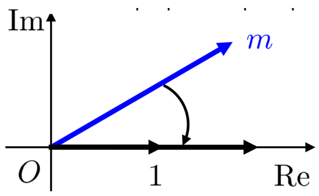
$$m \in \mathcal{C}\,,\quad \delta \in \mathcal{C} \qquad (1 - m\delta) = 0 \;\Rightarrow\; |\delta| = \frac{1}{|m|}$$

$$\mu_\delta(m) = \frac{1}{\min\{|\delta|\;|1 - m\delta = 0\}} = |m|$$

$$m = re^{j\phi}\,,\quad \delta = \frac{1}{r}e^{-j\phi} \qquad (1 - m\delta) = 0 \;\Rightarrow\; |\delta| = \frac{1}{|r|} = \frac{1}{|m|}$$

$$\mu_\delta(m) = |m|$$

$$m = 0 \qquad \mu_\Delta(m) = 0$$



(Ref 1, pp. 309-312)

Mathematical Properties of μ

○ **μ of Full Block**

$$M \in \mathcal{C}^{n\times n}\,,\quad \Delta = \left[\,\Delta\,\right] \in \mathcal{C}^{n\times n} \qquad M = U\Sigma V^H,\quad \Delta = (1/\sigma_1)v_1 u_1^H$$

$$\mu_\Delta(M) = \bar{\sigma}(M)$$ Equal to the maximal singular value in the absence of the structure

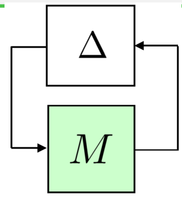
$$\begin{bmatrix} \det(I - M\Delta) = \det(I - U\Sigma V^H v_1 u_1^H/\sigma_1) = 1 - u_1^H U\Sigma V^H v_1/\sigma_1 = 0 \\ u_1, v_1 : \text{1st columns of } U, V,\;\; \sigma_1 = \bar{\sigma}(M) \;\because\; \det(I - AB) = \det(I - BA) \end{bmatrix}$$

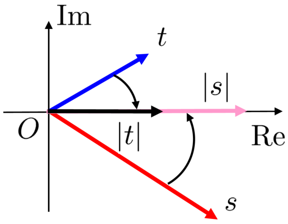
(Ref 1, pp. 309-312)

Mathematical Properties of μ

- **Special Case of 2×2 Matrices**

$$M=\begin{bmatrix} t & t \\ s & s \end{bmatrix}, \Delta=\begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} \quad \mathrm{rank}(M)=1$$





$$M\Delta=\begin{bmatrix} t & t \\ s & s \end{bmatrix}\begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}=\begin{bmatrix} t \\ s \end{bmatrix}\begin{bmatrix} \delta_1 & \delta_2 \end{bmatrix}=: \tilde{M}\tilde{\Delta}$$

$$\det(I-M\Delta)=\det(I-\tilde{M}\tilde{\Delta})=\det(I-\tilde{\Delta}\tilde{M})=1-t\delta_1-s\delta_2$$

$$|\delta_1|=|\delta_2|=\frac{1}{|t|+|s|}; \quad 1-t\delta_1-s\delta_2=0 \quad \Rightarrow \quad \mu_\Delta(M)=|t|+|s|$$

- **Scaling**

$$\mathbf{\Delta}=\begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} \Rightarrow \mu\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}=\mu\begin{bmatrix} m_{11} & dm_{12} \\ \frac{1}{d}m_{21} & m_{22} \end{bmatrix}$$

Mathematical Properties of μ

- **Example:**

$$M=\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}=\begin{bmatrix} 0.894 & 0.447 \\ -0.447 & 0.894 \end{bmatrix}\begin{bmatrix} 3.162 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix}^H$$

1) Full perturbation

$$\Delta=\frac{1}{3.162}\begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}\begin{bmatrix} 0.894 & -0.447 \end{bmatrix}=\begin{bmatrix} 0.200 & -0.100 \\ 0.200 & -0.100 \end{bmatrix}$$

$$\bar{\sigma}(\Delta)=\frac{1}{3.162}=0.316 \qquad \mu_\Delta(M)=\frac{1}{\bar{\sigma}(\Delta)}=3.162$$

$$\mu_\Delta(M)=\bar{\sigma}(M)=\sqrt{2\cdot|2|^2+2\cdot|-1|^2}=3.1623$$

2) Diagonal perturbation

$$\Delta=\begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}=\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}\in\mathbf{\Delta}$$

$$\bar{\sigma}(\Delta)=0.333=\frac{1}{\mu_\Delta(M)} \qquad \mu_\Delta(M)=|2|+|-1|=3$$

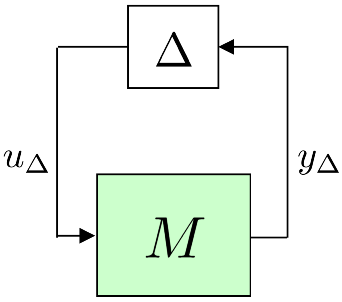
Stability of Closed Loop Systems

Unstructured Uncertainties

$$\Delta = \left[\,\Delta\,\right] = \begin{bmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{bmatrix}$$

Full block

$$\|\Delta\|_\infty \le 1$$



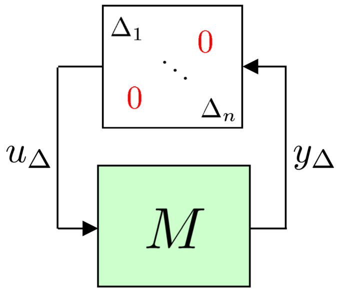
$$\|M\|_\infty < 1 \;\; (\bar{\sigma}(M) < 1)$$

(Small gain theorem)

Structured Uncertainties

$$\Delta = \mathrm{diag}\{\Delta_i\} = \begin{bmatrix} \Delta_1 & & & 0 \\ & \ddots & & \\ & & \Delta_i & \\ 0 & & & \ddots \end{bmatrix}$$

$$\|\Delta\|_\infty \le 1$$



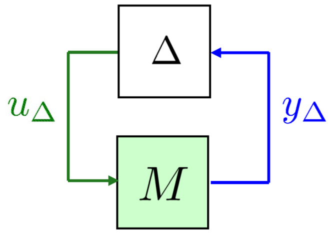
$$\mu_\Delta(M) < 1$$

(Ref 1, pp. 301-303)

Stability of MΔ-Structure

$$\Delta = \mathrm{diag}\{\Delta_i\} = \begin{bmatrix} \Delta_1 & & & 0 \\ & \ddots & & \\ & & \Delta_i & \\ 0 & & & \ddots \end{bmatrix}$$

(Structured)

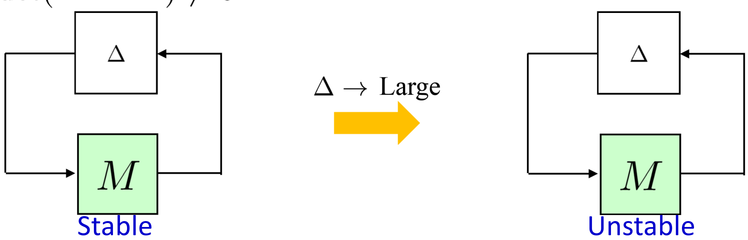


$$\begin{cases} y_\Delta = M u_\Delta \\ u_\Delta = \Delta y_\Delta \end{cases} \Rightarrow \; y_\Delta = M\Delta y_\Delta \;\rightarrow\; (I - M\Delta) y_\Delta = 0$$

$$\det(I - M\Delta) = 0 \;?$$

$$\Delta = 0 : \det(I - M \cdot 0) = 1 \neq 0 \;\; \text{(Stable)}$$

$\det(I - M\Delta) \neq 0$ $\det(I - M\Delta) = 0\,!$



Stable Unstable

(Ref 1, p. 301)

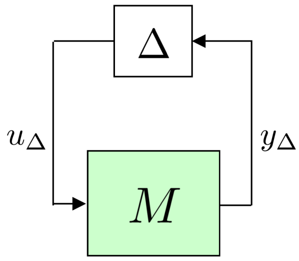
Continue

$$\Delta = \begin{bmatrix} \Delta_1 & & & 0 \\ & \ddots & & \\ & & \Delta_i & \\ 0 & & & \ddots \end{bmatrix}$$

$$\Delta = \begin{bmatrix} \Delta_i & \\ & \Delta_j \end{bmatrix}$$

$$\Delta = \begin{bmatrix} \Delta_i & \\ & \Delta_j \end{bmatrix}$$

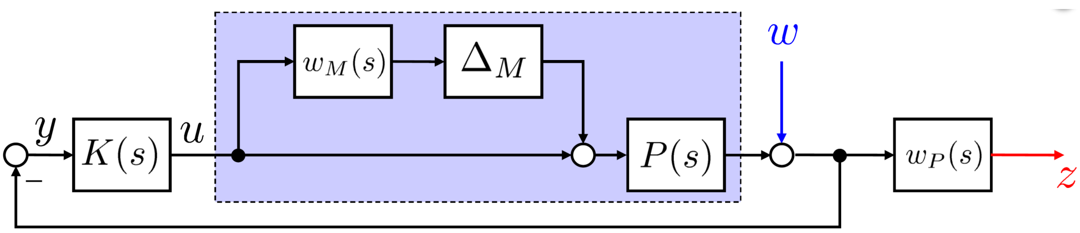
$$\bar{\sigma}(\Delta) = \max_i \{\bar{\sigma}(\Delta_i)\}$$



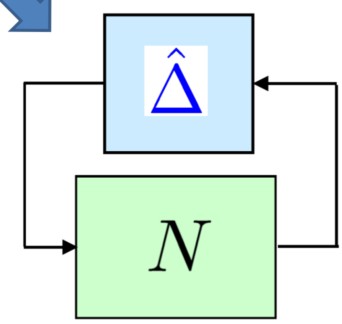
- Determine the size of Δ which makes the system unstable, when we make the structured uncertainty Δ large gradually.
- Find the smallest structured Δ satisfying $\det(I - M\Delta) = 0$ and determine the size $\bar{\sigma}(\Delta)$.

$$\mu(M)^{-1} = \min_{\Delta}\{\bar{\sigma}(\Delta) | \det(I - M\Delta) = 0 \text{ for Structured } \Delta\}$$

Robust Performance in SISO Systems



$$N = \begin{bmatrix} -w_M T & -w_M KS \\ w_P PS & w_P S \end{bmatrix}$$

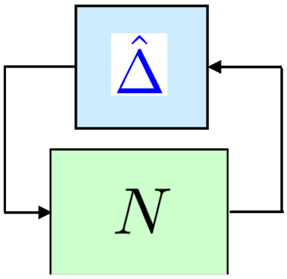


$$\mu_{\hat{\Delta}}(N) < 1$$

(Ref 1, p. 322)

Continue

$$\mu_{\hat{\Delta}}(N) < 1$$



$$N = \begin{bmatrix} -w_M T & -w_M KS \\ w_P PS & w_P S \end{bmatrix}$$

$$\mu \begin{bmatrix} -w_M T & -w_M KS \\ w_P SP & w_P S \end{bmatrix} \begin{matrix} \times P \\ \times P^{-1} \end{matrix} = \mu \begin{bmatrix} -w_M T & -w_M T \\ w_P S & w_P S \end{bmatrix} = |w_M T| + |w_P S| < 1, \ \forall \omega$$

$$(T = PK(1+PK)^{-1} = PKS)$$

Structured Uncertainty

for $\Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}$, $\mu \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \mu \begin{bmatrix} m_{11} & dm_{12} \\ \frac{1}{d} m_{21} & m_{22} \end{bmatrix}$, $\mu \begin{bmatrix} t & t \\ s & s \end{bmatrix} = |t| + |s|$

(Ref 1, p. 322)

Bounds of Structured Singular Value (μ)

$$\max_{U \in \mathcal{U}} \rho(MU) \leq \mu_\Delta(M) \leq \min_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1})$$

[bounds, muinfo] = **mussv**(M, blk, option)

Output argument:

bounds: Upper and lower bounds of μ

muinfo: Information on these bounds

Input argument:

M: Frequency transfer matrix

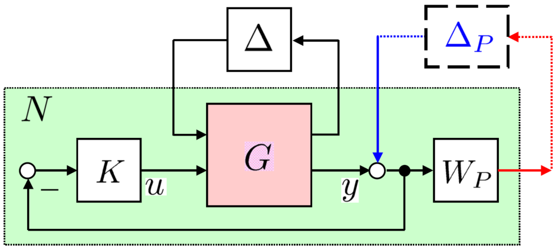
blk: Structure of uncertainty

- Information on the bound can be extracted by the command mussvextract

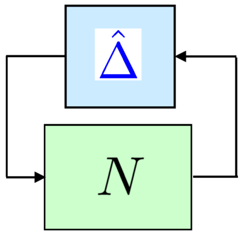
Option:

‘**a**’: These bounds are computed to the maximal accuracy of LMI solver
‘**f**’: Only the upper bound is roughly computed
‘**U**’: Only the upper bound is computed
‘**x**’: The lower bound is roughly computed

μ-Analysis for Robust Performance



$$\mu_{\hat{\Delta}}(N) < 1$$

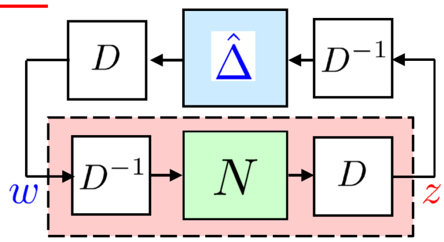


Since it is computationally hard to obtain μ, the bounds are employed:

$$\max_{U \in \mathcal{U}} \rho(NU) \leq \mu_{\hat{\Delta}}(N) \leq \min_{D \in \mathcal{D}} \bar{\sigma}(DND^{-1})$$

$$\min_{D \in \mathcal{D}} \bar{\sigma}(DND^{-1}) < 1$$

D-Scaled maximum singular value



Example: Spinning Satellite

Inverse-based Controller and H∞ Controller: Frequency Response

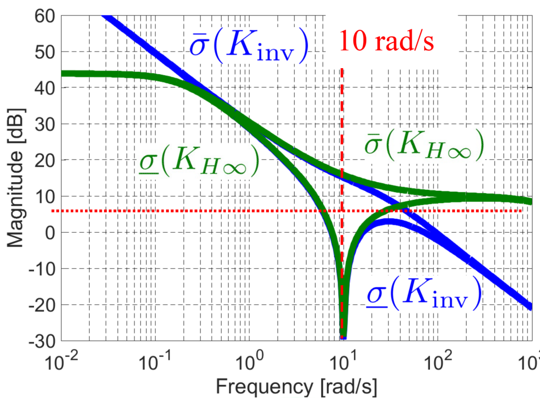
$$K_I(s) = P^{-1}(s) \begin{bmatrix} \frac{900}{s(s+30)} & 0 \\ 0 & \frac{900}{s(s+30)} \end{bmatrix}$$

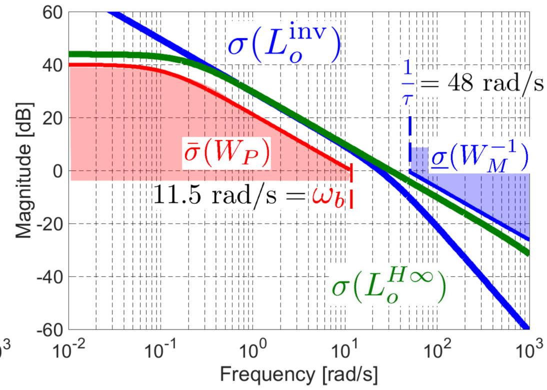
hinfsyn

NS ✓ **H∞ Controller**

NP ✓ $\|W_P S\|_\infty < 1$

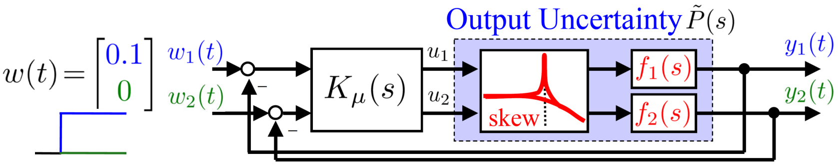
RS ✓ $\|W_M T\|_\infty < 1$

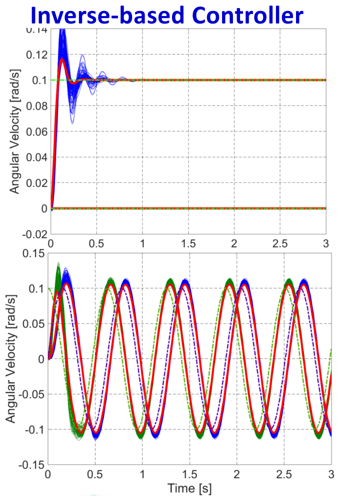




Example: Spinning Satellite

Inverse-based Controller and
H∞ Controller: Time Response



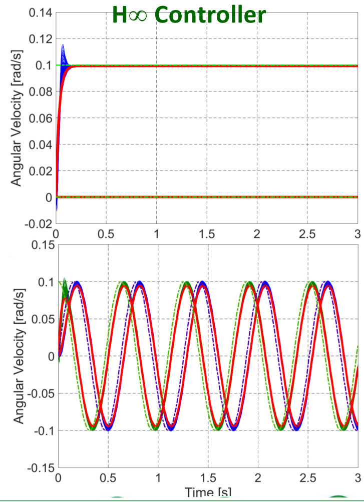


$$w(t) = \begin{bmatrix} 0.1\sin(\omega t) \\ 0.1\cos(\omega t) \end{bmatrix}$$

$$\omega = 10 \text{ rad/s}$$

- - - Reference
—— Nominal Model
— — Perturbed Model

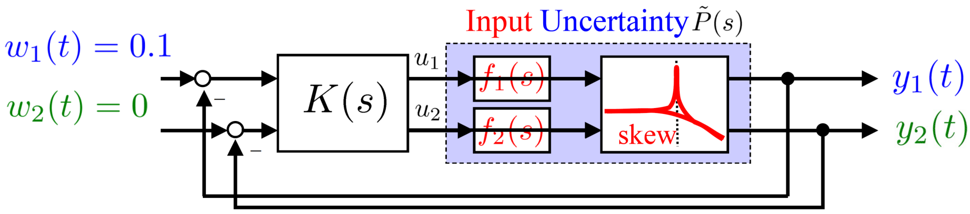
NS √
NP √
RS √

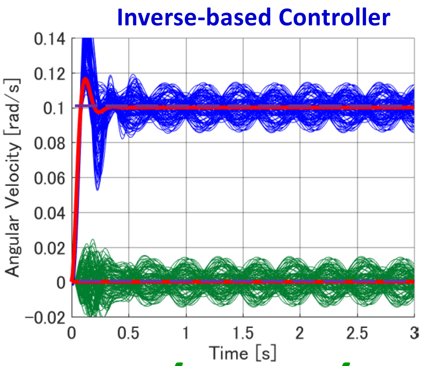


NS √
NP √
RS √

Example: Spinning Satellite

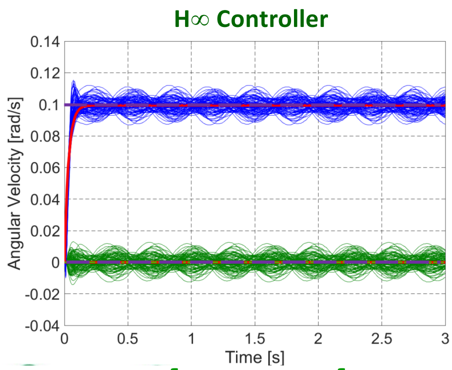
Inverse-based Controller and
H∞ Controller: Time Response
for Input Uncertainty





NS √ RS √ (Input)
NP √ RP X

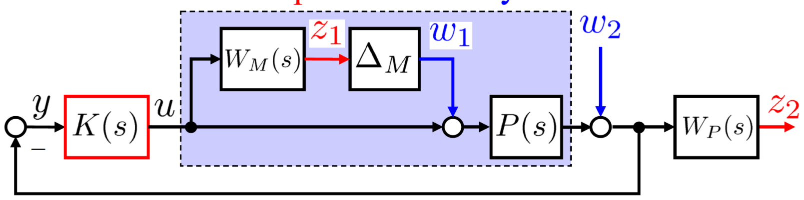
- - - Reference
—— Nominal Model
— — Perturbed Model



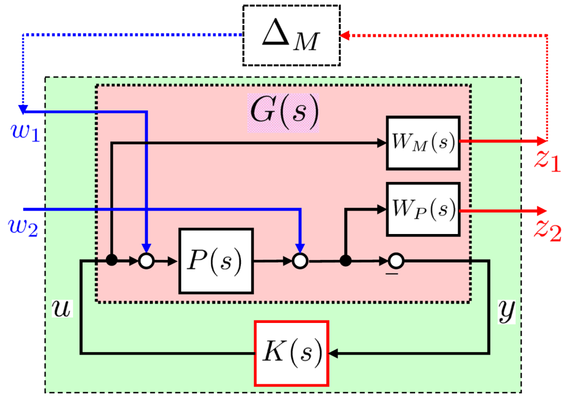
NS √ RS √ (Input)
NP √ RP X

Example: Spinning Satellite

Input Uncertainty



- **Generalized Plant**



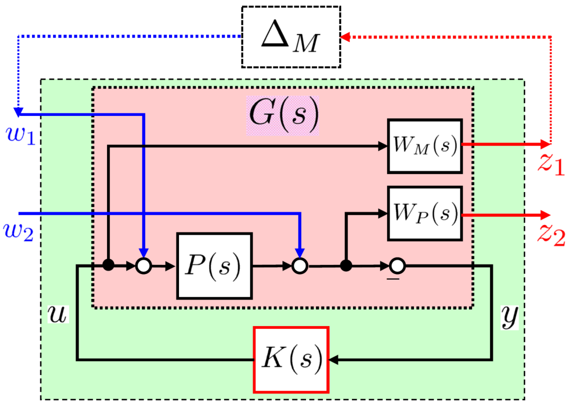
$$z = F_l(G,K)w$$

$$F_l(G,K) = \begin{bmatrix} -W_M T_i & -W_M K S_o \\ W_P S_o P & W_P S_o \end{bmatrix}$$

Example: Spinning Satellite

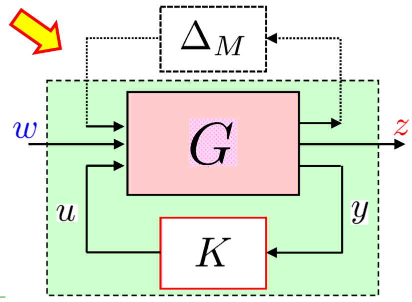
MATLAB Command

```
%Generalized Plant%
systemnames = 'Pnom WP WM';
inputvar = '[w1(2); w2(2); u(2)]';
outputvar = '[WM;WP;-w2-Pnom]';
input_to_Pnom = '[u+w1]';
input_to_WP = '[w2+Pnom]';
input_to_WM = '[u]';
G = sysic;
%with Structured Uncertainty%
unc1 = ultidyn('unc1',[1 1]);
unc2 = ultidyn('unc2',[1 1]);
unc = [unc1 0; 0 unc2];
Gunc = lft(unc,G);
```



$$\Delta_M = \begin{bmatrix} \delta_{M1} & 0 \\ 0 & \delta_{M2} \end{bmatrix}$$

Structured Uncertainty

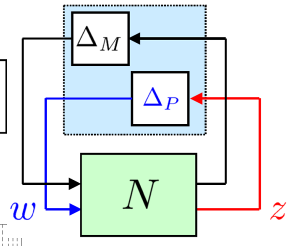


Example: Spinning Satellite

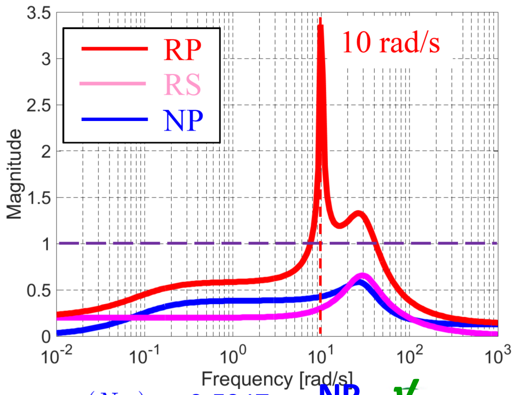
mussv

μ-Analysis of Inverse-based and H∞ Control Designs

$$N = F_l(G,K) = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} -W_M T_i & -W_M K S_o \\ W_P S_o P & W_P S_o \end{bmatrix}$$



Inverse-based Controller

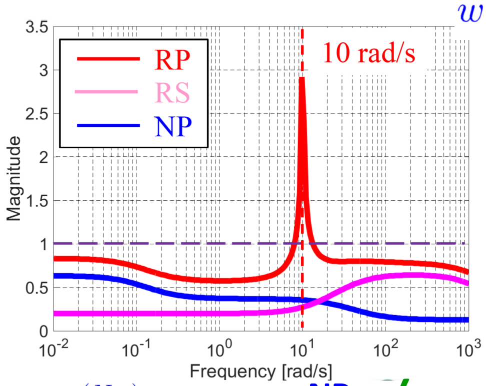


$\mu_{\Delta_P}(N_{22})$ = 0.5847 **NP** √

$\mu_{\Delta_M}(N_{11})$ = 0.6569 **RS** √ (Input)

$\mu_{\hat{\Delta}}(N)$ = 3.3644 **RP** X

H∞ Controller



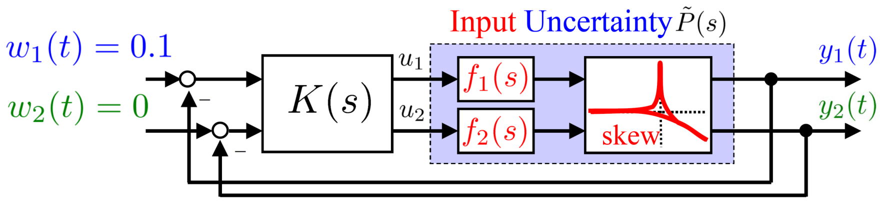
$\mu_{\Delta_P}(N_{22})$ = 0.6331 **NP** √

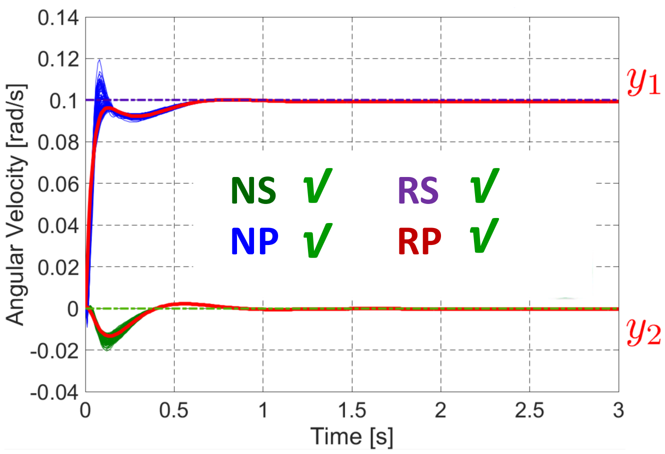
$\mu_{\Delta_M}(N_{11})$ = 0.6416 **RS** √ (Input)

$\mu_{\hat{\Delta}}(N)$ = 2.9159 **RP** X

Example: Spinning Satellite

Time Responses of Closed-loop system with μ-Controller



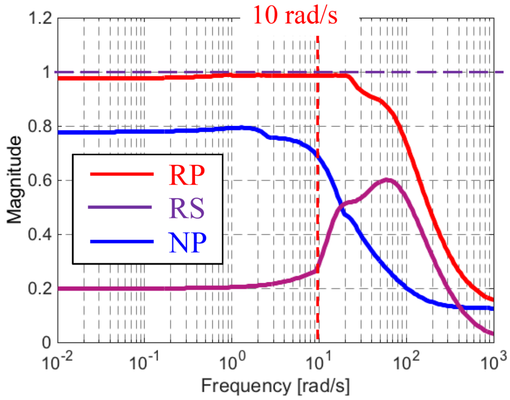


Example: Spinning Satellite

μ-Analysis of μ Controller

mussv

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} -W_M T_i & -W_M K S_o \\ W_P S_o P & W_P S_o \end{bmatrix}$$



$\mu_{\Delta_P}(N_{22})$ = 0.7936 **NP** √

$\mu_{\Delta_M}(N_{11})$ = 0.6001 **RS** √

$\mu_{\hat{\Delta}}(N)$ = 0.9898 **RP** √

MATLAB Command

```
Blk_unc = [1 1;1 1];
Blk_per = [2 2];
Blk = [Blk_unc; Blk_per];
%%%
w = logspace(-2,2,200);
Nf = frd(N,w);
%%% mu for NP %%%
Nnp = Nf(3:4,3:4);
[MuBnds,MuInfo] = mussv(Nnp,Blk_per,'c');
muNP = MuBnds(:,1);
[muNPinf,muNPw] = norm(muNP,inf);
muNPinf
 %%% mu for RS %%%
Nrs = Nf(1:2,1:2);
[MuBnds,MuInfo] = mussv(Nrs,Blk_unc,'c');
muRS = MuBnds(:,1);
[muRSinf,muRSw] = norm(muRS,inf);
muRSinf
%%% mu for RP %%%
[MuBnds,MuInfoRP] = mussv(Nf,Blk,'c');
muRP = MuBnds(:,1);
[muRPinf,muRPw] = norm(muRP,inf);
muRPinf
%%%
figure; sigma(muNP,muRS,muRP)
```


μ-Synthesis: D-K Iteration

[k, cl, bnd, info] = **dksyn**(G, nmeas, ncont, option)

- **Input argument:**

  G: Generalized Plant
  nmeas: Number of measurement outputs
  ncont: Number of control inputs

- **Output argument:**

  k: Controller
  cl: Closed-loop system which consists of G and K
  bnd: Upper bound of μ
  info: Information of Iteration

- Note: use dksynOptions/dkitopt to create **option**

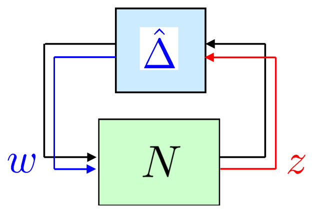
μ-Synthesis: D-K Iteration

$$\min_K \max_\omega \mu_{\hat{\Delta}}\left(N(j\omega)\right)$$

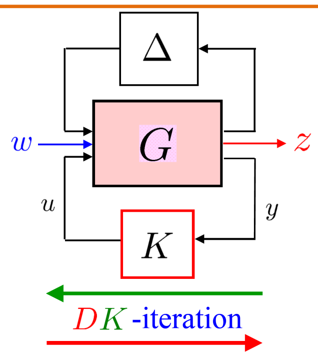
At present there is no direct method to synthesize a μ -optimal controller.

$$\min_K \max_\omega \min_{D_\omega \in \mathcal{D}} \bar{\sigma}\left(D_\omega N(j\omega) D_\omega^{-1}\right)$$

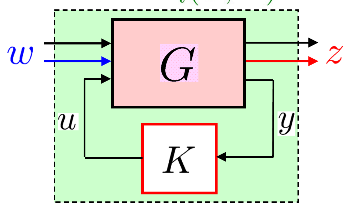
$$\min_K \left(\min_{D \in \mathcal{D}} \|DN(j\omega)D^{-1}\|_\infty \right)$$ Scaled $H\infty$ Control



(D-Step)
μ-analysis
Structured Uncertainty



$z = F_l(G, K)w$
$N = F_l(G, K)$



(K-Step)
H_∞-synthesis

(Ref 1, p. 328)

Robust Stability for Block-Diagonal Perturbations

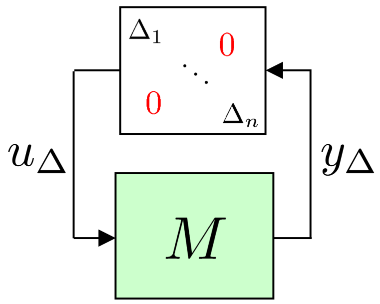
Theorem: Assume that the nominal system $M(s)$ and the perturbations $\Delta(s)$ are stable. Then the $M\Delta$-system is stable for all perturbation with

$$\bar{\sigma}(\Delta(j\omega)) \le 1, \ \forall\omega \iff \mu_\Delta(M) < 1, \ \forall\omega$$

or

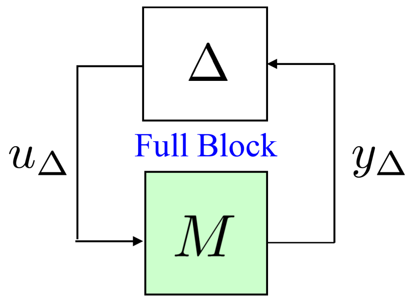
$$\bar{\sigma}(\Delta(j\omega)) \le \frac{1}{\beta}, \ \forall\omega \iff \mu_\Delta(M) < \beta, \ \forall\omega$$

Structured



$\mu_\Delta(M) < 1 \ \forall\omega$

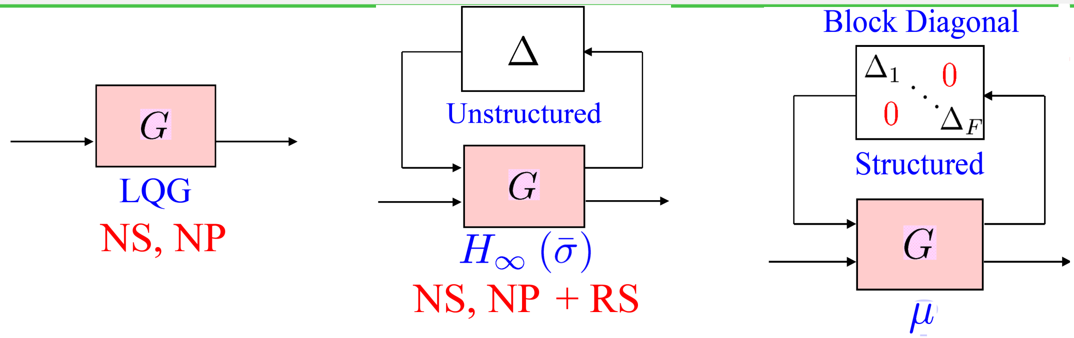
Unstructured



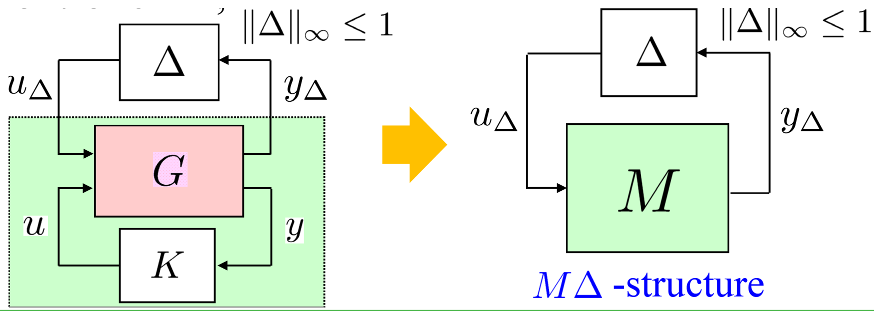
$\mu_\Delta(M) = \bar{\sigma}(M) < 1 \ \forall\omega$

(Ref 1, p. 314)

Structured Uncertainty



- Feedback System with Structured Uncertainties



Bounds on Structured Singular Value μ

$$\boxed{\max_{U\in\mathcal{U}} \rho(MU)} \le \mu_\Delta(M) \le \boxed{\min_{D\in\mathcal{D}} \bar{\sigma}(DMD^{-1})}$$

- **Example:**

$$M = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$$ Special case of 2×2 Matrices — mussv

(i) blk = $\Delta = \begin{bmatrix} \delta_{11} & 0 \\ 0 & \delta_{22} \end{bmatrix}$

(ii) blk = $\Delta = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix}$

Structured (Block Diagonal)

MATLAB Command

```
M = [2 2; -1 -1];
blk = [1 0;1 0]; % structured
[bounds,muinfo] = mussv(M,blk);
```

Result

bounds = 3.0000 3.0000

$$\mu_\Delta(M) = |2| + |-1| = 3$$

Unstructured (Full Block)

MATLAB Command

```
M = [2 2; -1 -1];
blk = [2 2]; % unstructured
[bounds,muinfo] = mussv(M,blk);
```

Result

bounds = 3.1623 3.1623

$$\mu_\Delta(M) = \bar{\sigma}(M) = \sqrt{10} = 3.1623$$

Mathematical Properties of μ

- **Lemma:** $\mu_\Delta(M) = \max_{\Delta \in \mathbf{B\Delta}} \rho(M\Delta)$ $\quad \mathbf{B\Delta} = \{\Delta \in \mathbf{\Delta} : \bar{\sigma}(\Delta) < 1\}$

  where $\rho(A) := \max_i |\lambda_i(A)|$ denotes spectral radius of matrix A .

- **Properties:**

2. $\mu_\Delta(M) = \rho(M)$ for $\mathbf{\Delta} = \{\delta I : \delta \in \mathcal{C}\}$ (Repeated Scalar Perturbation)
3. $\mu_\Delta(M) = \bar{\sigma}(M)$ for $\mathbf{\Delta} = \mathcal{C}^{n\times n}$ (Full-block Perturbation)
4. $\rho(M) \le \mu_\Delta(M) \le \bar{\sigma}(M)$, $\{\delta I : \delta \in \mathcal{C}\} \subset \mathbf{\Delta} \subset \mathcal{C}^{n\times n}$
6. $D\Delta = \Delta D$, $D \in \mathcal{D}, \Delta \in \mathbf{\Delta}$.

   Then, $\mu_\Delta(DM) = \mu_\Delta(MD)$, $\mu_\Delta(M) = \mu_\Delta(DMD^{-1})$

   $\mathcal{U} = \{U \in \mathbf{\Delta} : UU^H = I_n\}$ (Unitary Matrix)

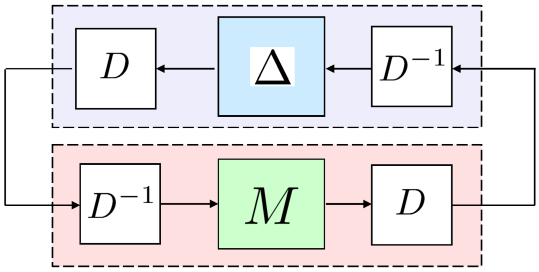
   $\mathcal{D} = \{\mathrm{diag}(d_1 I_{m_1}, \cdots, d_{F-1} I_{m_{F-1}}, I_{m_F}) : d_i \in \mathcal{R}, d_i > 0\}$
7. Upper Bound and Lower Bound (Reduction of Conservatism)

$$\max_{U \in \mathcal{U}} \rho(MU) \le \mu_\Delta(M) \le \min_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1})$$

(Ref 1, pp. 309-312)

Computation of Upper Bound of μ

$$\mu_\Delta(M) \le \min_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1}) < \beta$$



$$\bar{\sigma}(DMD^{-1}) < \beta \Leftrightarrow (DMD^{-1})^H DMD^{-1} < \beta^2 I$$

$$\Leftrightarrow M^H D^H DM - \beta^2 D^H D < 0$$

$$\inf_{D \in \mathcal{D}} \min_{\beta} \{\beta : M^H DM - \beta^2 D < 0\}$$

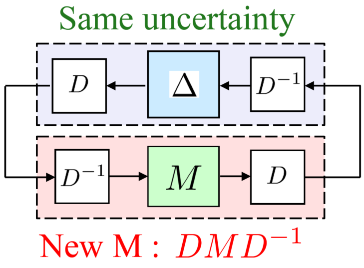
It may be solved using LMI

(Ref 1, p. 336)

Upper and Lower Bounds of μ

$$\max_{U\in\mathcal{U}} \rho(MU) \le \mu_\Delta(M) \le \min_{D\in\mathcal{D}} \bar\sigma(DMD^{-1})$$

- **Example:** $M=\begin{bmatrix} t & t \\ s & s\end{bmatrix}, \Delta=\begin{bmatrix}\delta_1 & 0\\ 0 & \delta_2\end{bmatrix}$



Lower bound $U=\mathrm{diag}\{e^{j\phi},1\}$

$$\max_{U\in\mathcal{U}} \rho(MU) = \max_\phi \left|\mathrm{tr}\left(\begin{bmatrix} te^{j\phi} & t\\ se^{j\phi} & s\end{bmatrix}\right)\right| = \max_\phi |te^{j\phi}+s|$$

Upper bound $D=\mathrm{diag}\{d,1\}$ $\left(\|A\|_F=\sqrt{\sum_i \sigma_i^2(A)}\right)$

$$\bar\sigma(DMD^{-1}) = \|DMD^{-1}\|_F = \sqrt{|t|^2+|dt|^2+|s/d|^2+|s|^2}$$

$$\min_{D\in\mathcal{D}} \bar\sigma(DMD^{-1}) = \min_d \sqrt{|t|^2+|dt|^2+|s/d|^2+|s|^2} = |s|+|t| = \mu_\Delta(M)$$

Structured Uncertainty $\Delta=\mathrm{diag}\{\delta_1 I,\cdots,\delta_S I,\Delta_1,\cdots,\Delta_F\}$

$2S+F\le 3 \Rightarrow \mu_{\hat\Delta}(N) = \min_{D\in\mathcal{D}} \bar\sigma(DND^{-1})$

Upper and Lower Bounds of μ

Repeated Scalar Complex Perturbation

$$\Delta=\begin{bmatrix}\delta & & 0\\ & \ddots & \\ 0 & & \delta\end{bmatrix} = \{\delta I : \delta\in\mathcal{C}\}$$

$$\mu_\Delta(M)=\rho(M)$$

Full-block Complex Perturbation

$$\Delta=\begin{bmatrix}\Delta\end{bmatrix}=\mathcal{C}^{n\times n}$$

$$\mu_\Delta(M)=\bar\sigma(M)$$

$$\rho(M)\le\mu_\Delta(M)\le\bar\sigma(M)$$

Structured Singular Value

- **Theorem (Upper Bound and Lower Bound):**

$$\max_{U\in\mathcal{U}} \rho(MU) \le \mu_\Delta(M) \le \min_{D\in\mathcal{D}} \bar\sigma(DMD^{-1})$$

$$\mathcal{U}=\{U\in\mathbf{\Delta} : UU^H=I_n\}$$

Unitary Matrix

$$\mathcal{D}=\{D : D\Delta=\Delta D\}$$

⇩

LMI

(Ref 1, pp. 309, 310)

Robust Performance in SISO Systems

NP: Nominal Performance

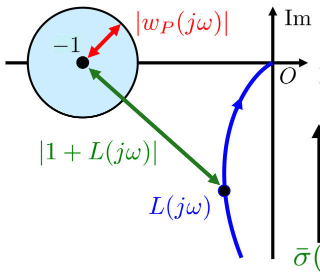
$$|w_P S| < 1 \ \forall \omega$$

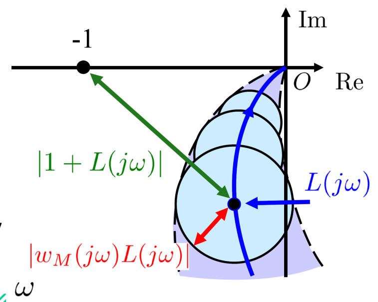
$$|w_P| < |1+L| \ \forall \omega$$

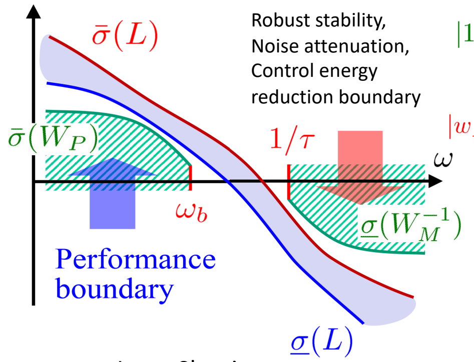
RS: Robust Stability

$$|w_M T| < 1 \ \forall \omega$$

$$|w_M L| < |1+L| \ \forall \omega$$







(Ref 1, p. 281)

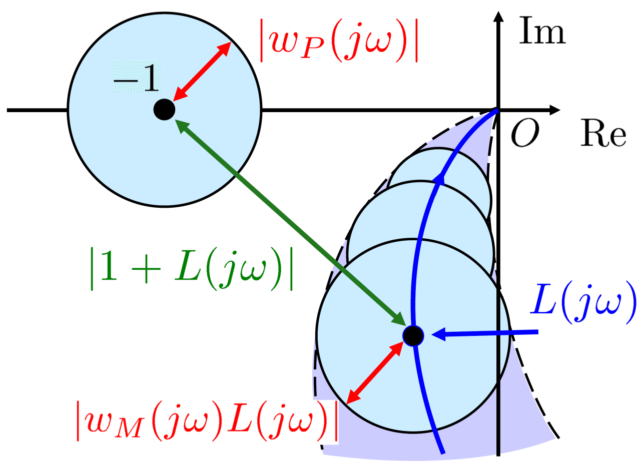
Loop Shaping

Robust Performance in SISO Systems

- RP: Robust Performance
  (beyond Loop Shaping)

$$|w_P S| + |w_M T| < 1, \ \forall \omega$$

$$|w_P| + |W_M L| < |1+L| \ \forall \omega$$



Thank You!