Robust Control Systems

Frequency Response Analysis (A Review)

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Contents

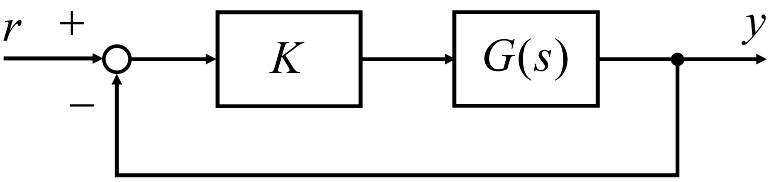
Reference

1. M. Fujita, **Lecture Notes on Feedback Control Systems**, Tokyo Institute of Technology, 2019.

2. H. Bevrani, **Lecture Notes on Linear Control Systems**, University of Kurdistan, 2022.

Root Locus

$$G(s)=\frac{N(s)}{D(s)}=\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$



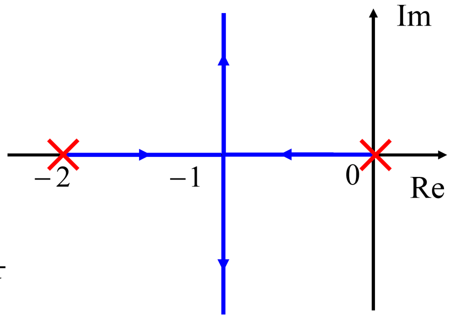
$$\frac{KG(s)}{1+KG(s)}=\frac{K\frac{N(s)}{D(s)}}{1+K\frac{N(s)}{D(s)}}=\frac{KN(s)}{D(s)+KN(s)}$$

$$\boxed{1+KG(s)=0}$$

Example: $G(s)=\frac{1}{s(s+2)}$

⇨ $D(s)+KN(s)=s(s+2)+K\cdot 1$
$=s^2+2s+K=0$

$K=0$	$s=0,-2$
$0<K<1$	$s=-1\pm\sqrt{1-K}$
$K=1$	$s=-1$
$K>1$	$s=-1\pm\sqrt{K-1}j$



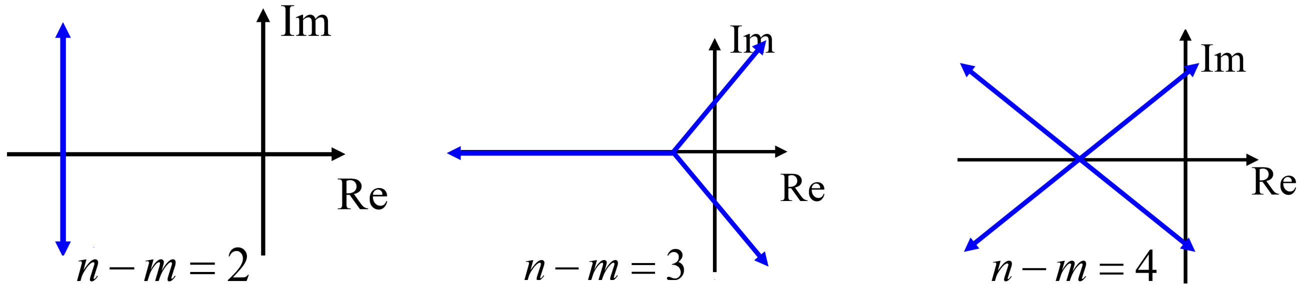
Root Locus

1

$$G(s)=\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

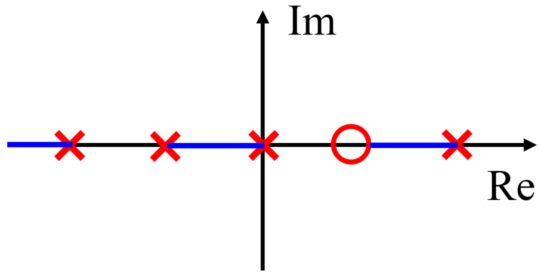
$$1+KG(s)=0 \quad \Rightarrow \quad G(s)=-\frac{1}{K}\rightarrow 0$$

2 $\frac{180^\circ+360^\circ l}{n-m}$ $\qquad$ $\frac{(p_1+p_2+\cdots+p_n)-(z_1+z_2+\cdots+z_m)}{n-m}$



Root Locus

3



4 $$\frac{d}{ds}\frac{1}{G(s)}=0$$

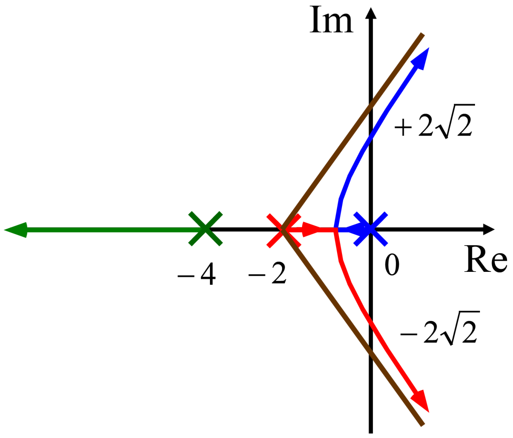
5

$$180^\circ-\sum_{i\neq j}\angle(p_j-p_i)+\sum_{i=1}^{m}\angle(p_j-z_i)$$

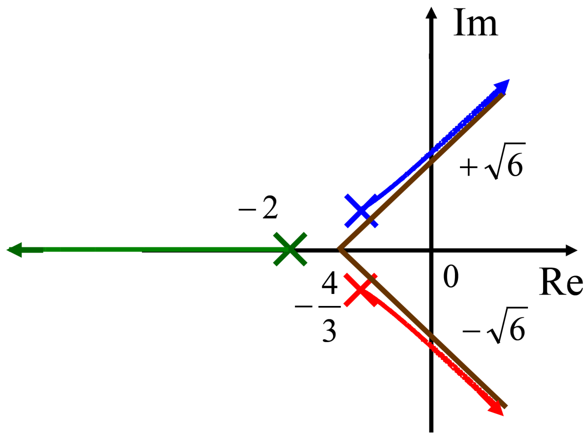
$$180^\circ+\sum_{i=1}^{n}\angle(z_j-p_i)-\sum_{i\neq j}\angle(z_j-z_i)$$

Root Locus

Examples:



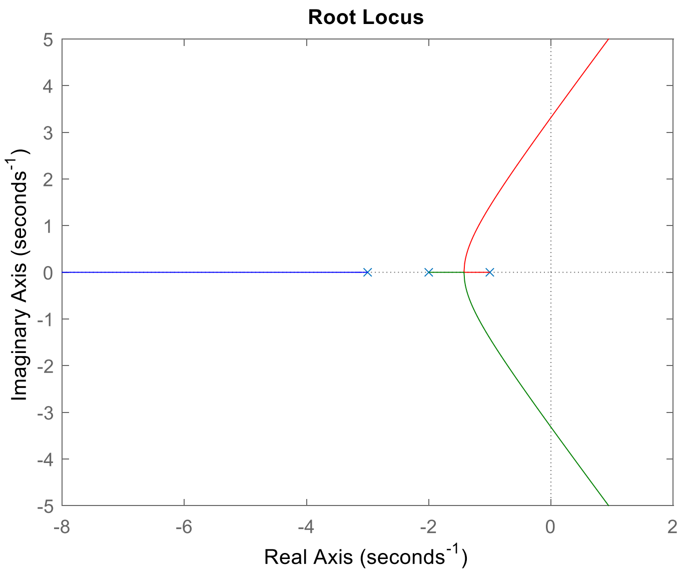
$$G(s)=\frac{1}{s(s+2)(s+4)}$$



$$G(s)=\frac{1}{(s+2)(s^2+2s+2)}$$

Example (Root Locus)

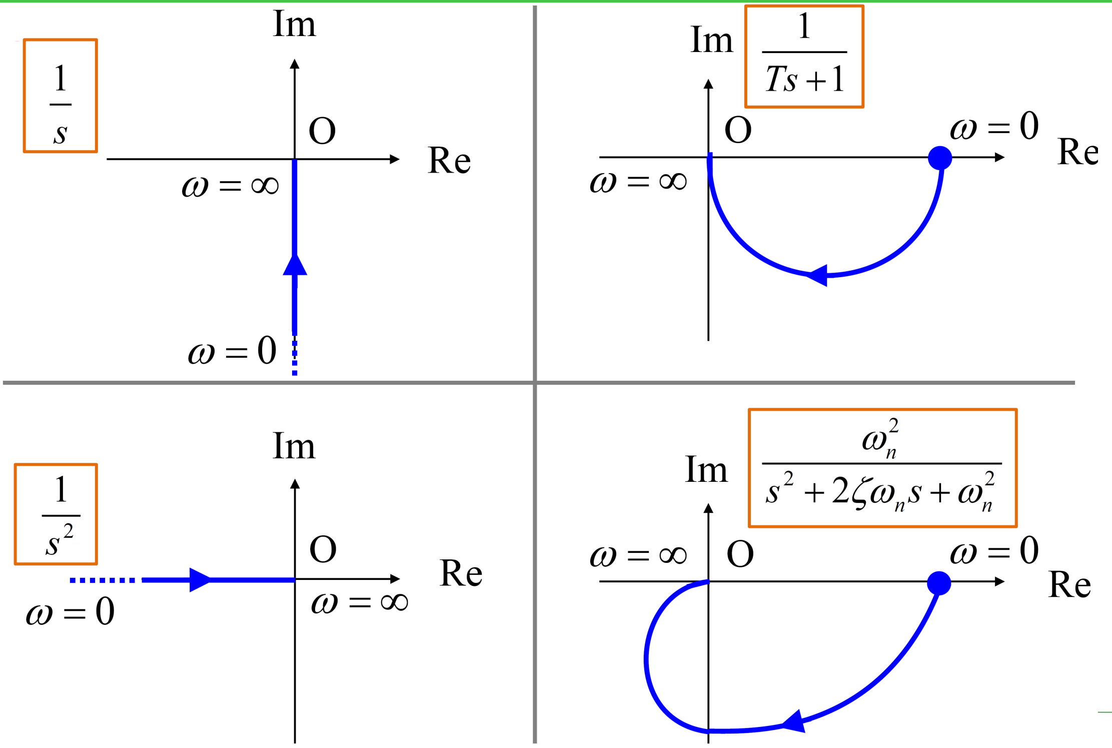
$$L(s)=P(s)K(s)\ =\frac{30}{(s+1)(s+2)(s+3)}$$



rlocus

```
num=[0 0 0 30];
den=conv(conv([1,1],[1,2]),[0,1,3]);
L=tf(num,den);
figure(2)
rlocus(L)
```

Vector Locus (Nyquist Graph)

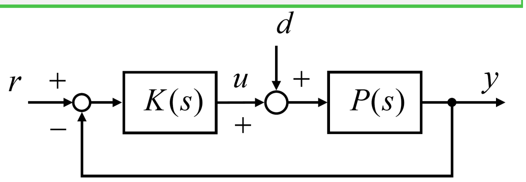


Nyquist

$$\underline{1 + P(s)K(s)} = 1 + \frac{N_P(s)}{D_P(s)} \cdot \frac{N_K(s)}{D_K(s)}$$

$$= \frac{D_P(s)D_K(s) + N_P(s)N_K(s)}{D_P(s)D_K(s)}$$

$$= \frac{(s - r_1)(s - r_2)\cdots(s - r_n)}{(s - p_1)(s - p_2)\cdots(s - p_n)}$$



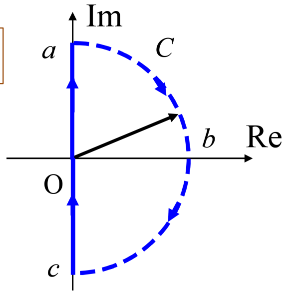
$Z = (\ \{r_1, r_2, \cdots, r_n\}$ **Closed-loop poles**

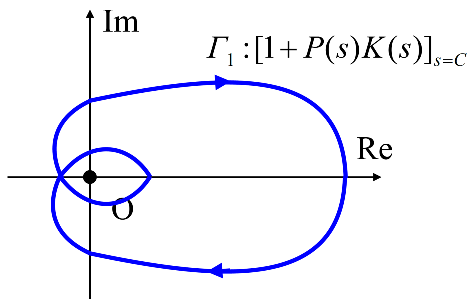
$\Pi = (\ \{p_1, p_2, \cdots, p_n\}$ **Open-loop poles**

$Z = N + \Pi$

Z=0 : Stable

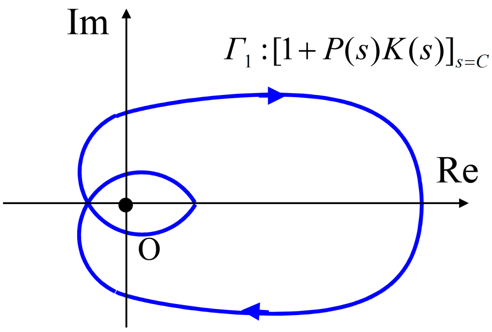
Z≠ 0: Unstable



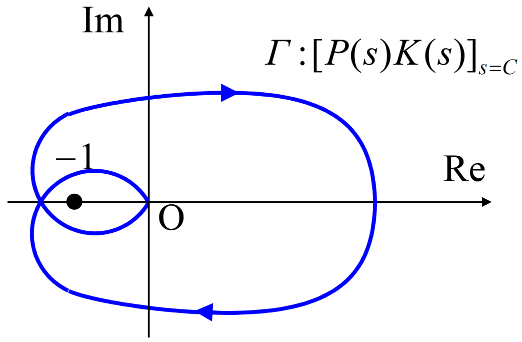


Nyquist

$\Gamma_1 \quad w = 1 + P(s)K(s)$ ⇨ **Nyquist:** $\Gamma \quad v = P(s)K(s)$

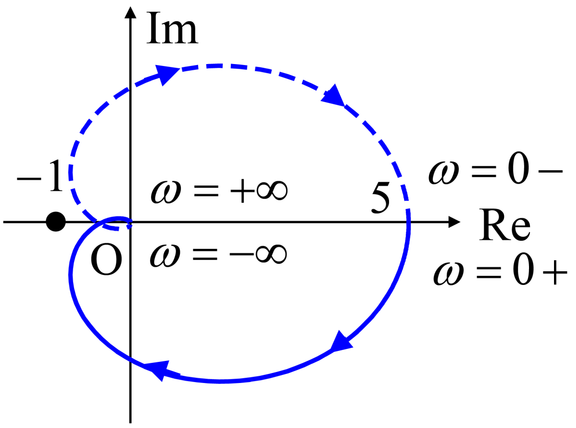


⇨



Nyquist: Example

1]



$$L(s) = P(s)K(s) = \frac{30}{(s+1)(s+2)(s+3)}$$

nyquist

```
num=[0 0 0 30];
den=conv(conv([1,1],[1,2]),[0,1,3]);
L=tf(num,den);
figure(1)
nyquist(L)
```

2] $N = 0$

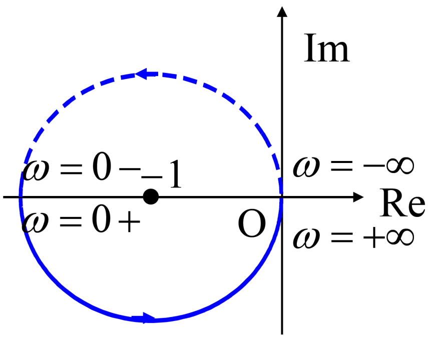
3] $\Pi = 0$

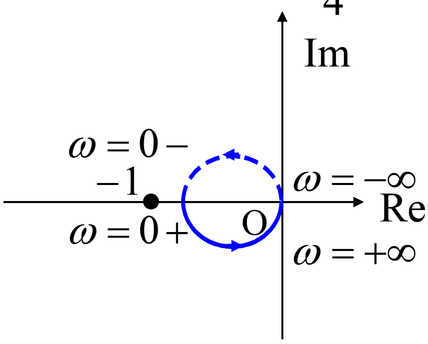
4] $Z = N + \Pi = 0$ **Stable**

Nyquist: Example

$L(s) = \frac{K}{s-1} \qquad K = 2, \ \frac{3}{4}$

1] (a) $K = 2$ | (b) $K = \frac{3}{4}$





2] $N = -1$ | $N = 0$

3] $\Pi = 1$ | $\Pi = 1$

4] $Z = N + \Pi = -1 + 1 = 0$ **Stable** | $Z = N + \Pi = 0 + 1 = 1 \neq 0$ **Unstable**

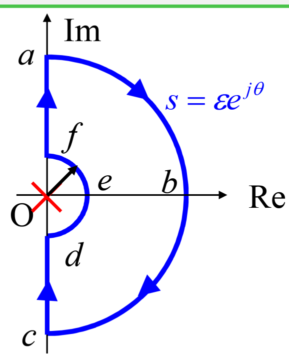
Nyquist: Example

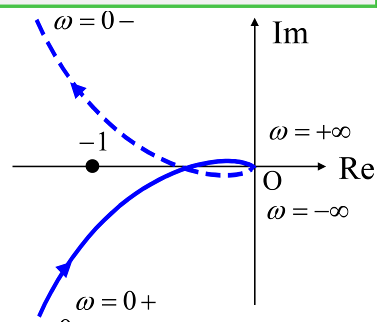
$$L(s) = \frac{K}{s(s+1)(s+2)}$$

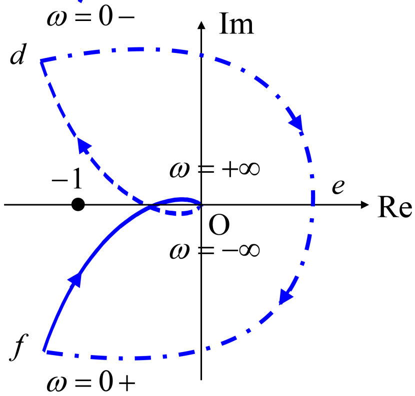
1] $\omega = 0$

$d \to e \to f$

$s = \varepsilon e^{j\theta} \ \ (\varepsilon \to 0, \ -90^\circ \le \theta \le 90^\circ)$







$$\lim_{\varepsilon \to 0} L(\varepsilon e^{j\theta}) = \lim_{\varepsilon \to 0} \frac{K}{\varepsilon e^{j\theta}(\varepsilon e^{j\theta}+1)(\varepsilon e^{j\theta}+2)} = \lim_{\varepsilon \to 0} \frac{K}{2\varepsilon} \underline{e^{-j\theta}}$$

2] $N = 0$

3] $\Pi = 0$

Stable

4] $Z = N + \Pi = 0$

Bode Diagram

$$G(j\omega) = \frac{1}{j\omega}$$

$$20\log|G(j\omega)| = 20\log\left|\frac{1}{j\omega}\right|$$

$$= 20\log\frac{1}{|\omega|} = -20\log|\omega|$$

$\omega = 0.1$

$-20\log 0.1 = -20\times(-1) = 20 \quad \text{dB}$

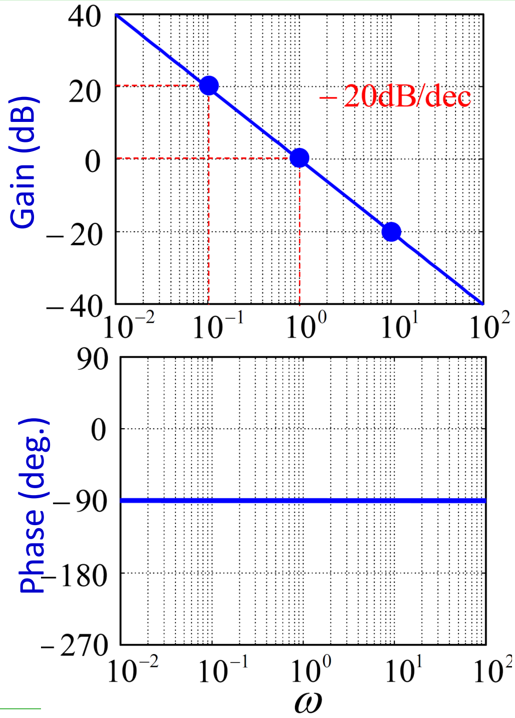
$\omega = 1$

$-20\log 1 = -20\times 0 = 0 \quad \text{dB}$

$\omega = 10$

$-20\log 10 = -20\times 1 = -20 \quad \text{dB}$

$$\angle G(j\omega) = \angle\frac{1}{j\omega} = \angle\frac{1}{j} = -\angle j = -90^\circ$$

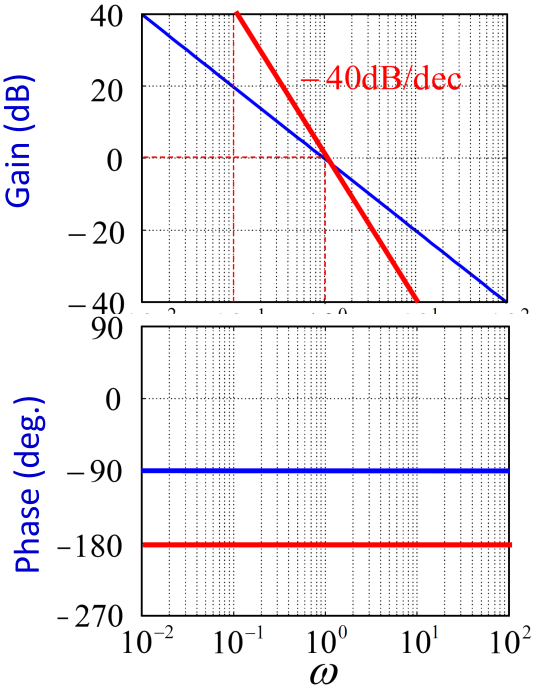


Bode Diagram

$$G(j\omega) = \frac{1}{(j\omega)^2}$$

$$20\log\frac{1}{|(j\omega)^2|} = 20\log\frac{1}{\omega^2} = -40\log|\omega|$$

$$\angle G(j\omega) = \angle\frac{1}{j^2} = \angle -1 = -180^\circ$$



Bode Diagram

$$G(j\omega)=\frac{1}{1+j\omega T}$$

$$20\log|G(j\omega)|=20\log\frac{1}{\sqrt{1+(\omega T)^2}}$$

$$\angle G(j\omega)=-\angle(1+j\omega T)=-\tan^{-1}(\omega T)$$

$$\omega T<<1 \qquad G(j\omega)\approx 1$$

$$\omega T>>1 \qquad G(j\omega)\approx\frac{1}{j\omega T}$$

$\omega T<<1 \quad 20\log|G|\approx 20\log 1=0$

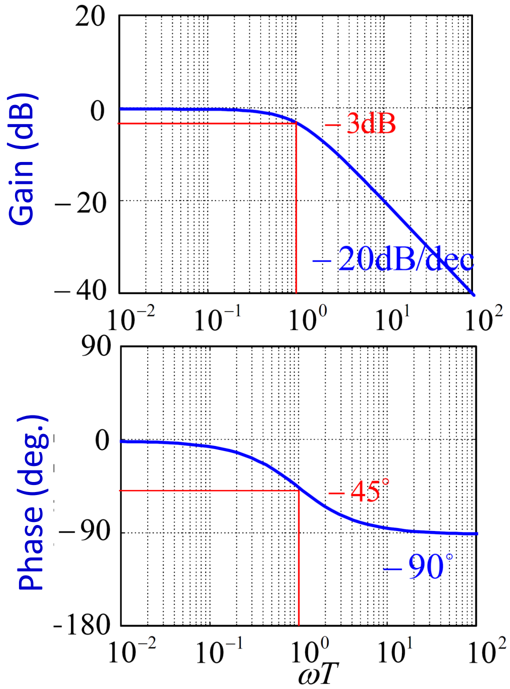
$\angle G=0^\circ$

$\omega T=1 \quad 20\log|G|=20\log\frac{1}{\sqrt{2}}=-3$ dB

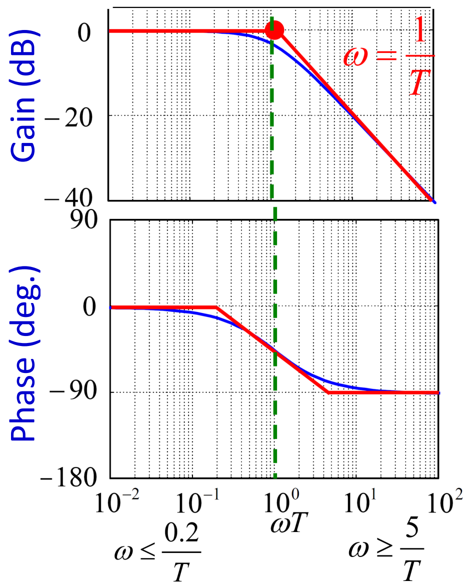
$\angle G=-45^\circ$

$\omega T>>1 \quad 20\log|G|\approx -20\log|\omega T|$ dB

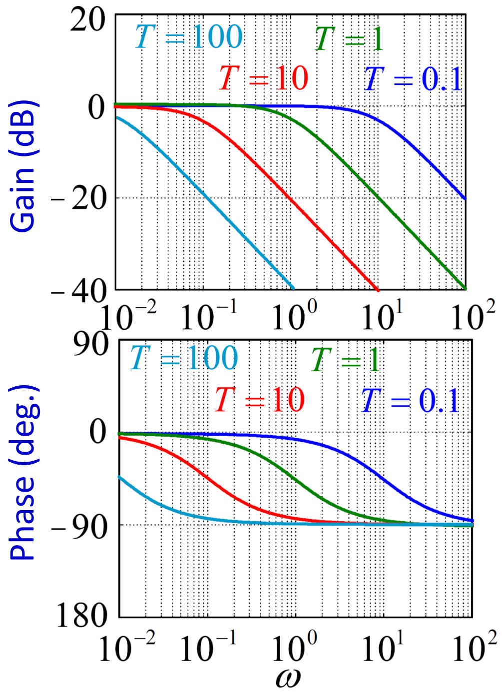
$\angle G\approx -90^\circ$



Bode Diagram

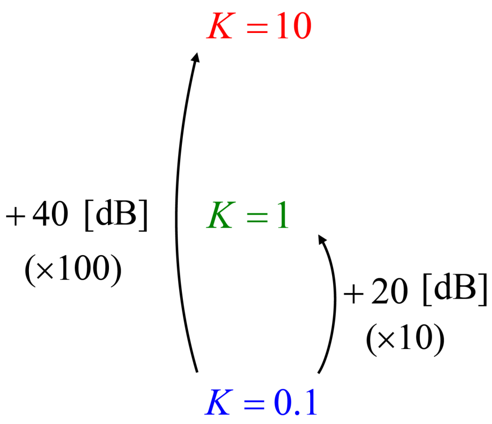


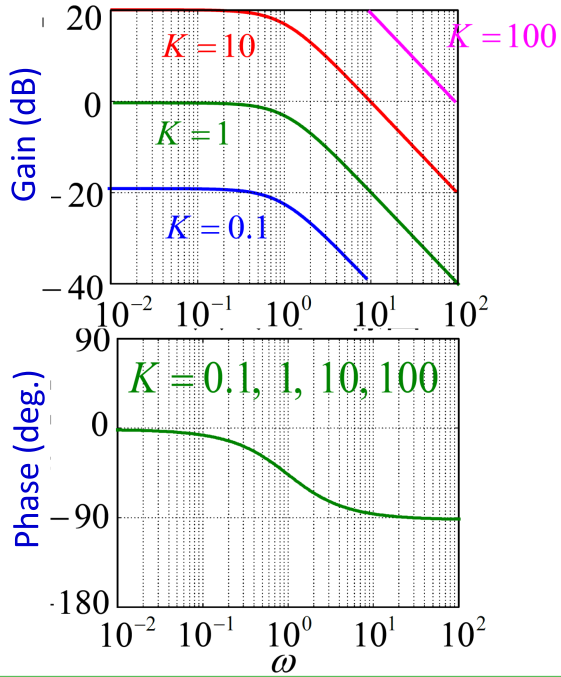
Approximated plot



Bode Diagram

$$G(j\omega) = \frac{K}{1 + j\omega T} \quad (T = 1)$$



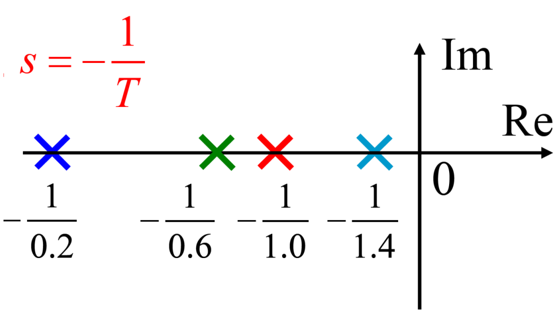


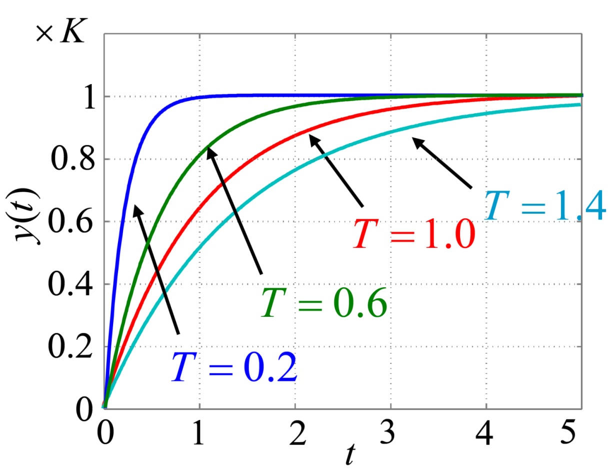
Root Impact on Step Response

$$G(s) = \frac{K}{Ts + 1} \Rightarrow y(t) = K(1 - e^{-t/T}) \Rightarrow$$

$$\lim_{t \to \infty} y(t) = \lim_{t \to \infty} K(1 - e^{-t/T}) = K$$

$$\left.\frac{dy}{dt}\right|_{t=0} = \left.\frac{K}{T} e^{-t/T}\right|_{t=0} = \frac{K}{T}$$





Bode Diagram: 2nd Order System

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \qquad (K=1)$$

$$G(j\omega)=\frac{\omega_n^2}{(j\omega)^2+2\zeta\omega_n\omega j+\omega_n^2}=\frac{1}{\left(j\frac{\omega}{\omega_n}\right)^2+2\zeta\frac{\omega}{\omega_n}j+1}$$

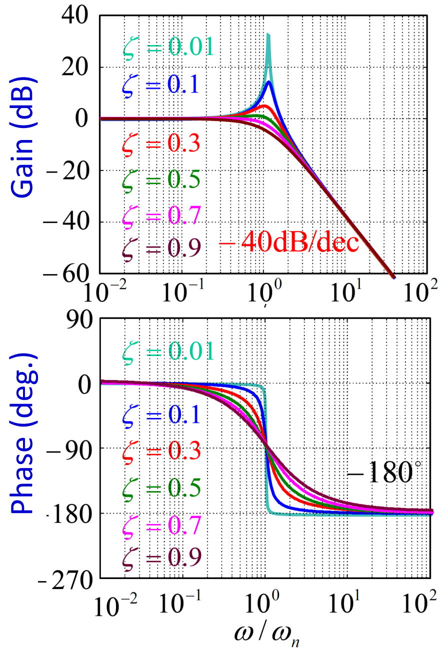
$$=\frac{1}{(j\Omega)^2+2\zeta\Omega j+1} \qquad \left(\Omega=\frac{\omega}{\omega_n}\right)$$

$$|G(j\omega)|=\frac{1}{\sqrt{(1-\Omega^2)^2+(2\zeta\Omega)^2}}$$

$$\angle G(j\omega)=-\angle([1-\Omega^2]\ +j[2\zeta\Omega])\ =-\tan^{-1}\frac{2\zeta\Omega}{1-\Omega^2}$$

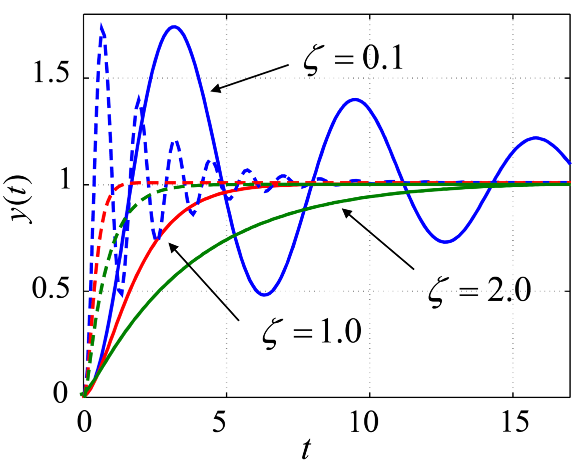
$\Omega<<1$	$20\log\lvert G\rvert\approx\ 20\log 1=0$ dB; $\angle G\approx 0^\circ$
$\Omega=1$	$20\log\lvert G\rvert=\ 20\log\left\lvert\frac{1}{2\zeta}\right\rvert$ dB; $\angle G=-90^\circ$
$\Omega>>1$	$20\log\lvert G\rvert\approx-40\log\lvert\Omega\rvert$ dB; $\angle G\approx-180^\circ$

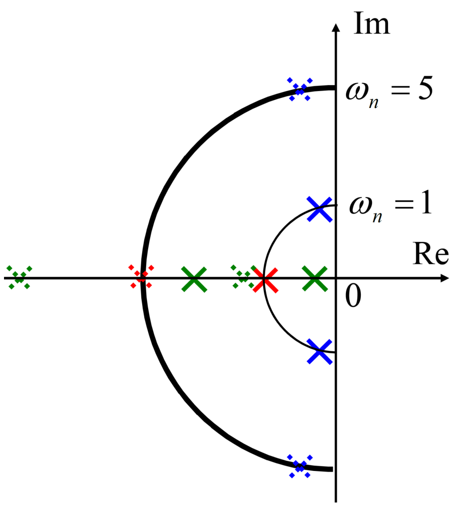
$\Omega<<1$	$G(j\omega)\approx 1$
$\Omega>>1$	$G(j\omega)\approx\frac{1}{-\Omega^2}$



Frequency/Step Response

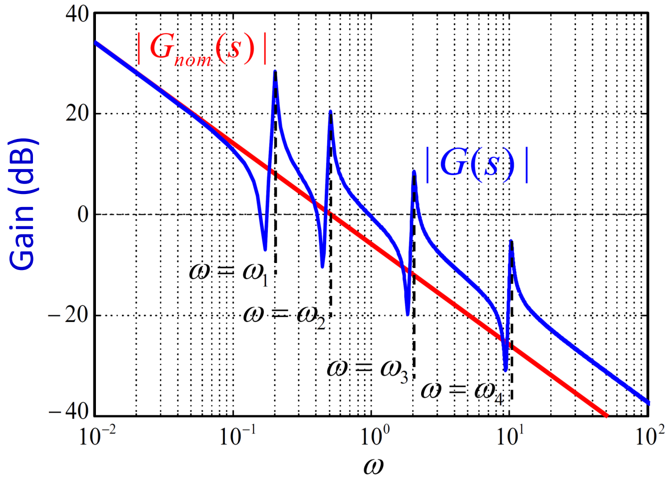
$$\begin{cases}\omega_n=1 & t=10\\ \omega_n=5 & t=2\end{cases}$$





Perturbed System

$$G(s) = \underbrace{\frac{0.5}{s}}_{G_{nom}(s)} + \underbrace{\sum_{i=1}^{4} \frac{0.2s}{s^2 + 2\zeta_i \omega_i s + \omega_i^2}}_{\text{Oscillation part}} \qquad \omega_1 = 0.2,\ \omega_2 = 0.5,\ \omega_3 = 2,\ \omega_4 = 10,\ \zeta_i = 0.02$$



Bode Diagram

$$G_1(s) = \frac{1+s}{s^2+s+1} \qquad G_2(s) = \frac{1-s}{s^2+s+1}$$

$$|G_1(j\omega)| = \frac{|1+j\omega|}{|(j\omega)^2 + j\omega + 1|} = \frac{|1+j\omega|}{|1-\omega^2+j\omega|} = \frac{\sqrt{1+\omega^2}}{|1-\omega^2+j\omega|}$$

$$= \frac{|1-j\omega|}{|1-\omega^2+j\omega|} = |G_2(j\omega)|$$

$$\angle G_1(j\omega) = \angle(1+j\omega) - \angle(1-\omega^2+j\omega)$$

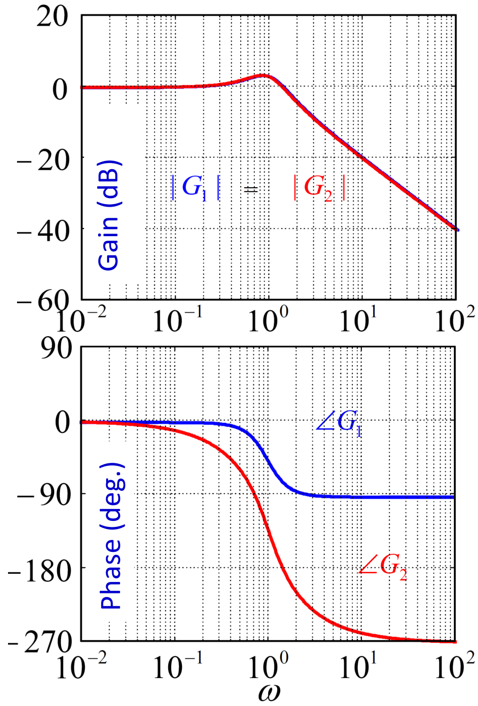
$$\angle G_2(j\omega) = \angle(1-j\omega) - \angle(1-\omega^2+j\omega)$$

$$\angle G_1(j\omega) \approx \angle j\omega - \angle(-\omega^2)$$

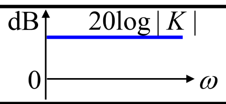
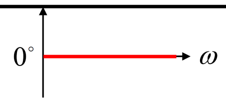
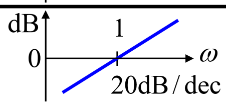
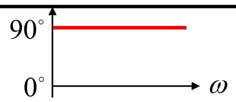
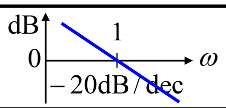
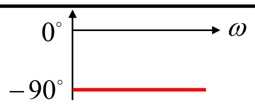
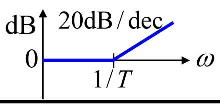
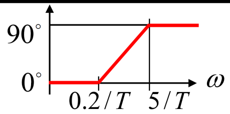
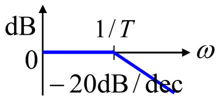
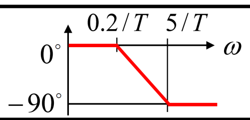
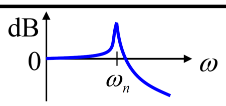
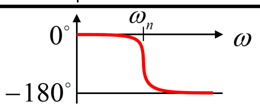
$$= +90^\circ - 180^\circ = -90^\circ$$

$$\angle G_2(j\omega) \approx \angle(-j\omega) - \angle(-\omega^2)$$

$$= -90^\circ - 180^\circ = -270^\circ$$

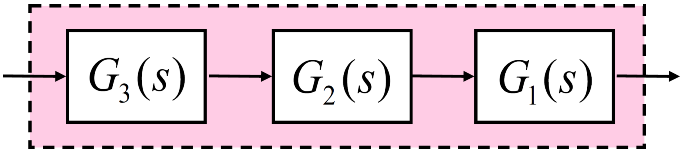


Bode Diagram: Summary

$G(s)$	Gain	Phase
K		
s		
$\frac{1}{s}$		
$Ts+1$		
$\frac{1}{Ts+1}$		
$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$		

Bode Diagram: Series Systems

$$G(s) = G_1(s)G_2(s)G_3(s)$$
$$G_i(j\omega) = r_i e^{j\theta_i} \qquad (i = 1 \sim 3)$$



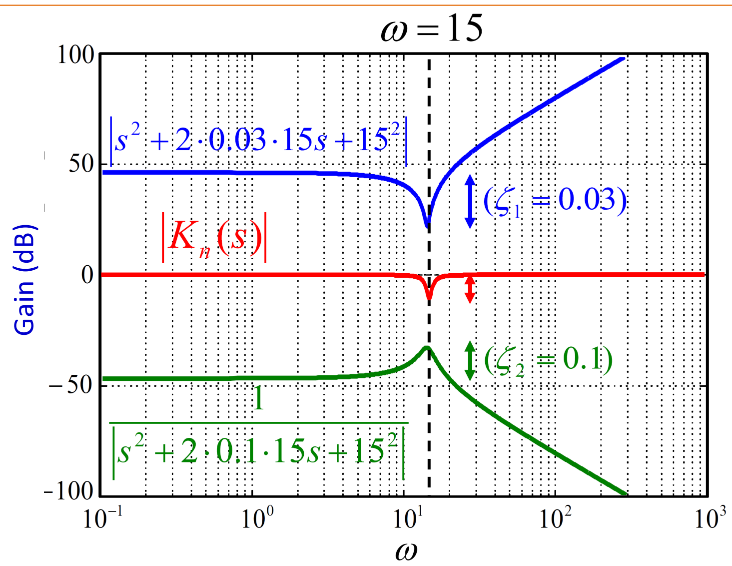
$$re^{j\theta} = (r_1 e^{j\theta_1})(r_2 e^{j\theta_2})(r_3 e^{j\theta_3}) = r_1 r_2 r_3 e^{j(\theta_1+\theta_2+\theta_3)}$$
$$r = r_1 r_2 r_3$$
$$\theta = \theta_1 + \theta_2 + \theta_3$$
$$\Rightarrow G(j\omega) = re^{j\theta} = r_1 r_2 r_3 e^{j(\theta_1+\theta_2+\theta_3)}$$

$$20\log|G(j\omega)| = 20\log r = 20\log(r_1 r_2 r_3)$$
$$= 20\log r_1 + 20\log r_2 + 20\log r_3$$
$$= \sum_{i=1}^{3} 20\log r_i = \sum_{i=1}^{3} 20\log|G_i(j\omega)|$$

$$\angle G(j\omega) = \theta = \theta_1 + \theta_2 + \theta_3$$
$$= \sum_{i=1}^{3}\theta_i = \sum_{i=1}^{3}\angle G_i(j\omega)$$

Example: Notch Filter

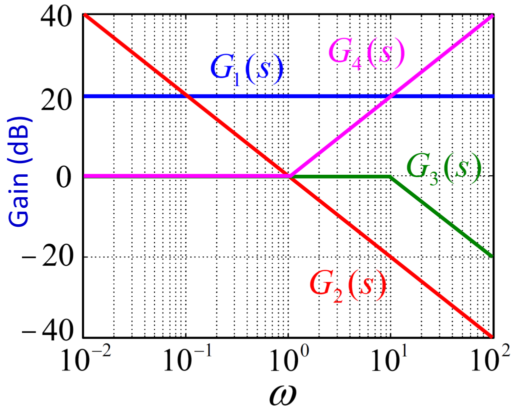
$$K_n(s) = \frac{s^2 + 2\cdot 0.03 \cdot 15s + 15^2}{s^2 + 2\cdot 0.1 \cdot 15s + 15^2} \qquad \begin{array}{l}(\zeta_1 = 0.03)\\ (\zeta_2 = 0.1)\end{array}$$

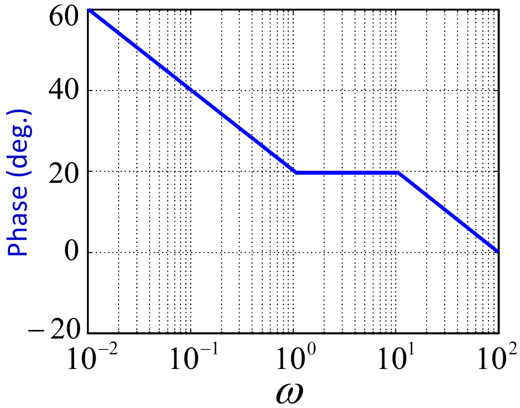


Example

$$G(s) = \frac{100(s+1)}{s(s+10)} = G_1(s)G_2(s)G_3(s)G_4(s)$$

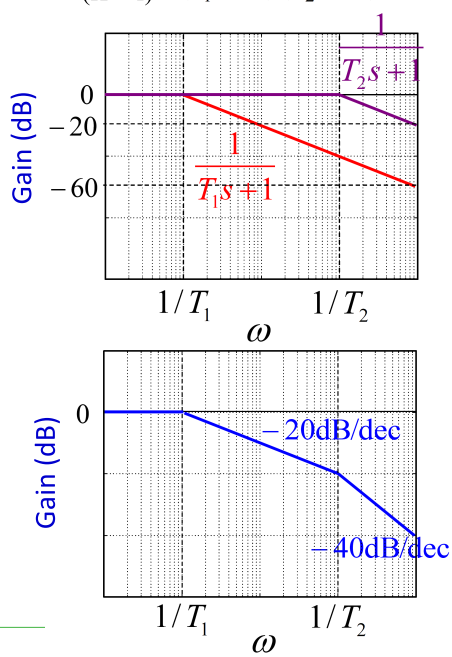
$$= 10 \cdot \frac{1}{s} \cdot \frac{1}{0.1s+1} \cdot (s+1)$$



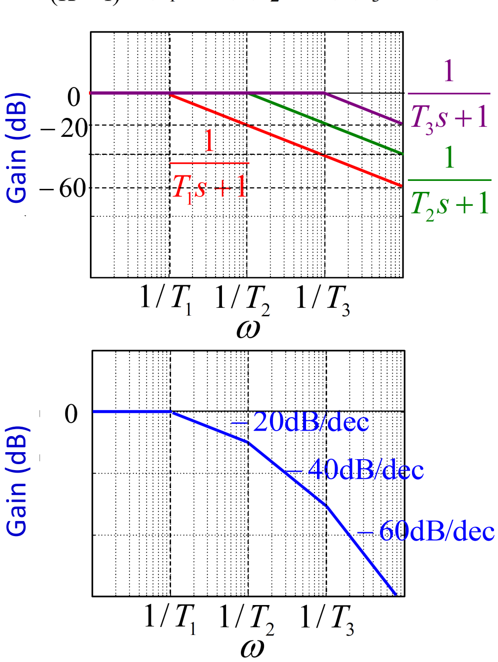


Example

$(K=1)\quad \dfrac{1}{(T_1 s+1)(T_2 s+1)}$

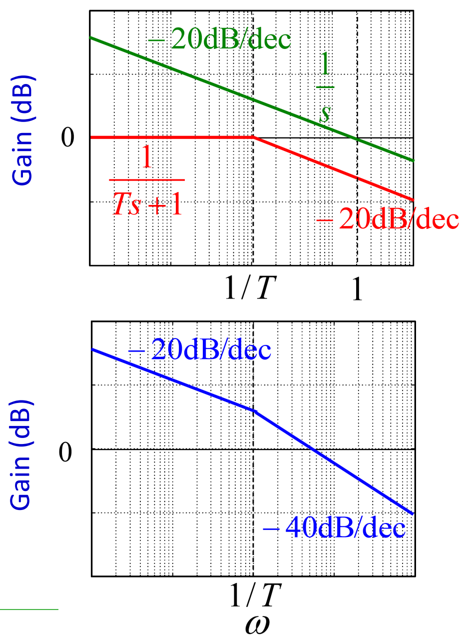


$(K=1)\quad \dfrac{1}{(T_1 s+1)(T_2 s+1)(T_3 s+1)}$

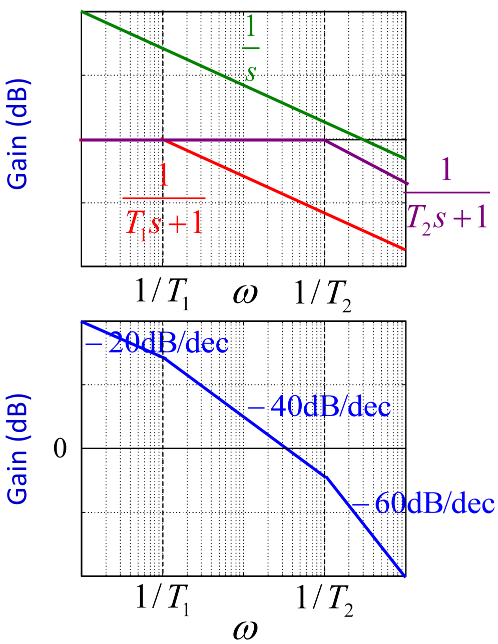


Example

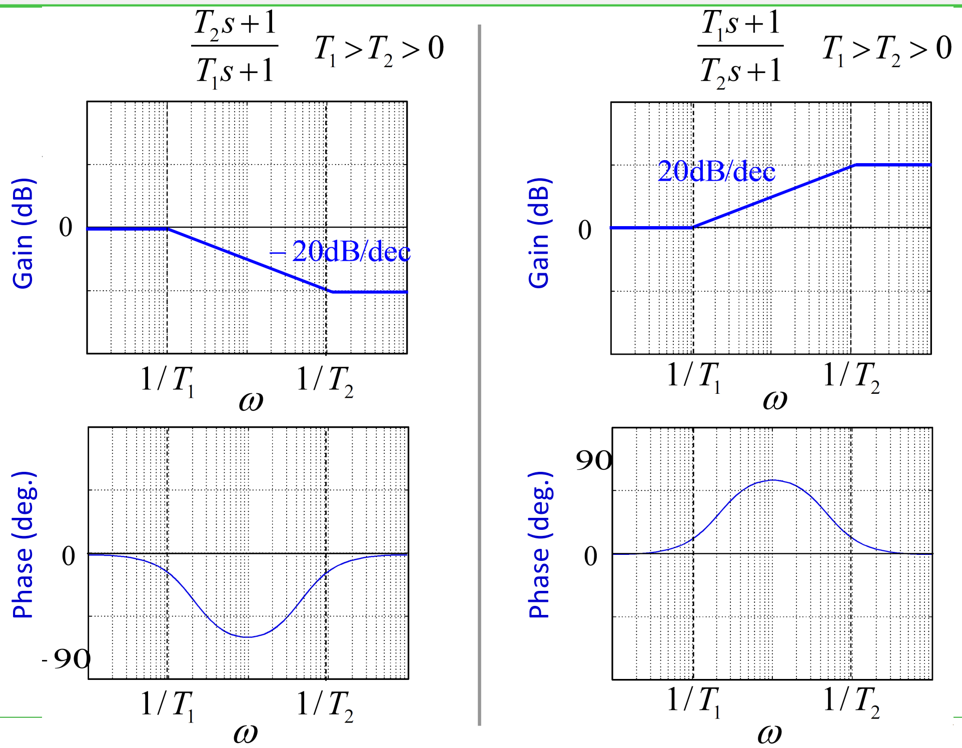
$\dfrac{1}{s(Ts+1)}$



$\dfrac{1}{s(T_1 s+1)(T_2 s+1)}$



Example



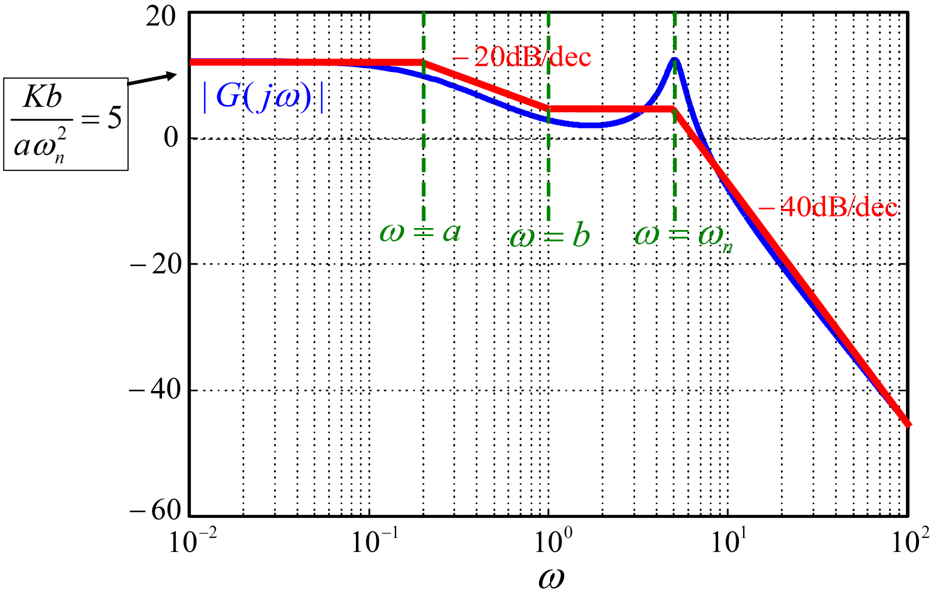
Example

$$G(s) = \frac{K(s+b)}{(s+a)(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$a = 0.2,\ b = 1$

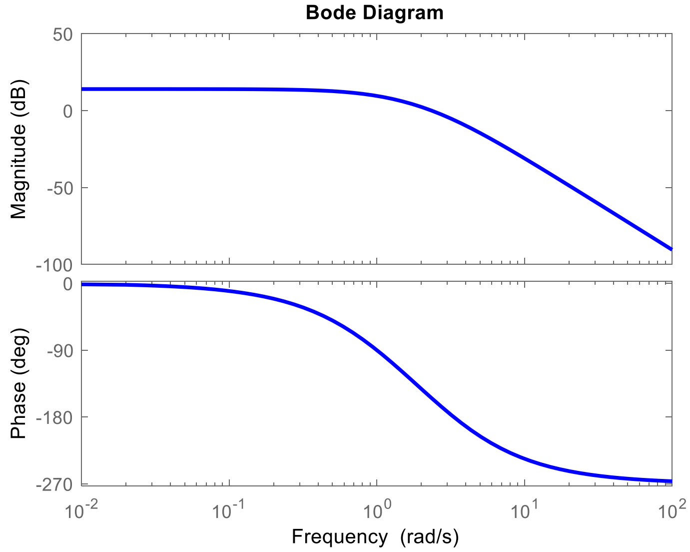
$\omega_n = 5,\ \zeta = 0.1$

$K = 25$



Continue (Bode)

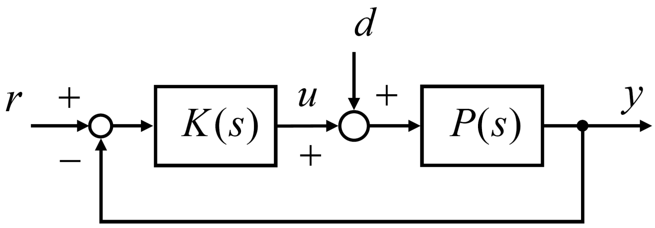
$$L(s) = P(s)K(s) = \frac{30}{(s+1)(s+2)(s+3)}$$



bode

```
num=[0 0 0 30];
den=conv(conv([1,1],[1,2]),[0,1,3]);
L=tf(num,den);
figure(2)
bode(L)
```


Feedback Systems



Gang of Four

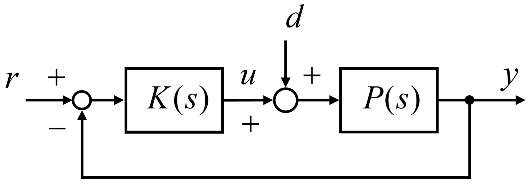
$$G_{ur}(s) = \frac{K(s)}{1+P(s)K(s)} \qquad G_{ud}(s) = \frac{-P(s)K(s)}{1+P(s)K(s)}$$

$$G_{yr}(s) = \frac{P(s)K(s)}{1+P(s)K(s)} \qquad G_{yd}(s) = \frac{P(s)}{1+P(s)K(s)}$$

Feedback: Characteristic Equation

$$P(s)=\frac{N_P(s)}{D_P(s)},\qquad K(s)=\frac{N_K(s)}{D_K(s)}$$

$$\phi(s):=D_P(s)D_K(s)+N_P(s)N_K(s)$$



$$G_{ur}=\frac{D_P(s)N_K(s)}{\phi(s)}\qquad G_{ud}=\frac{-N_P(s)N_K(s)}{\phi(s)}$$

$$G_{yr}=\frac{N_P(s)N_K(s)}{\phi(s)}\qquad G_{yd}=\frac{N_P(s)D_K(s)}{\phi(s)}$$

Example:

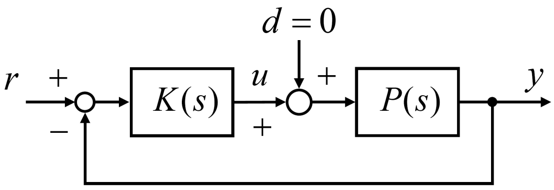
$$P(s)=\frac{1}{s-1}\qquad K(s)=\frac{s-1}{s}$$

$$\phi(s)=(s-1)\cdot s+1\cdot(s-1)\ =(s-1)(s+1)=0 \Rightarrow G_{yr}(s)=\frac{P(s)K(s)}{1+P(s)K(s)}=\frac{s-1}{(s-1)(s+1)}$$

Feedback

$$P(s)=\frac{1}{s-1},\quad K(s)=\frac{s-1}{s}\ =1-\frac{1}{s}$$

$$P(s)K(s)=\frac{1}{s-1}\cdot\frac{s-1}{s}\ =\frac{1}{s}$$

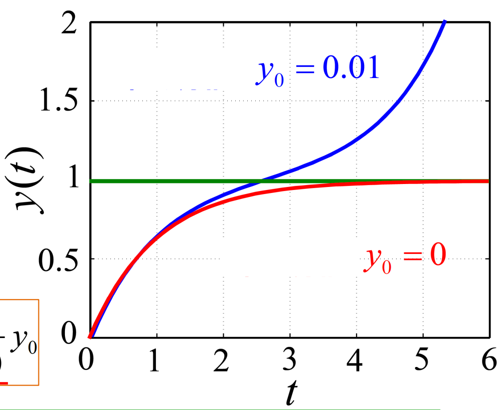


$$\Rightarrow\ y(s)=\frac{P(s)K(s)}{1+P(s)K(s)}\cdot r(s)=\frac{\frac{1}{s}}{1+\frac{1}{s}}\cdot r(s)\ =\frac{1}{s+1}\cdot r(s)$$

$$P(s)=\frac{y(s)}{u(s)}=\frac{1}{s-1}\ \Rightarrow\ \begin{array}{l}(s-1)y(s)=u(s)\\ \dot{y}(t)-y(t)=u(t)\end{array}$$

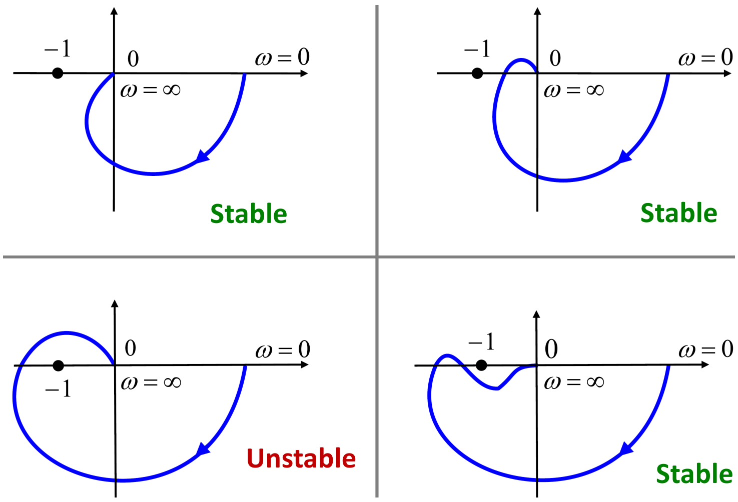
$$(sy(s)-y_0)-y(s)=u(s)\ \Rightarrow\ y(s)=\frac{1}{s-1}u(s)+\frac{1}{s-1}y_0$$

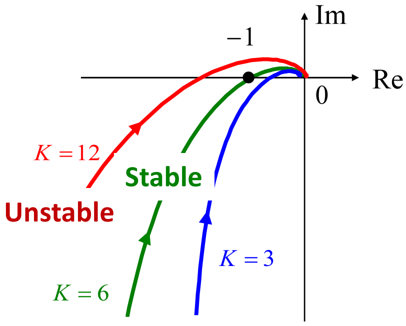
$$y(s)=\frac{1}{s-1}\cdot\frac{s-1}{s}(r(s)-y(s))+\frac{1}{s-1}y_0\ \Rightarrow\ y(s)=\frac{1}{s+1}r(s)+\frac{s}{(s+1)(s-1)}y_0$$



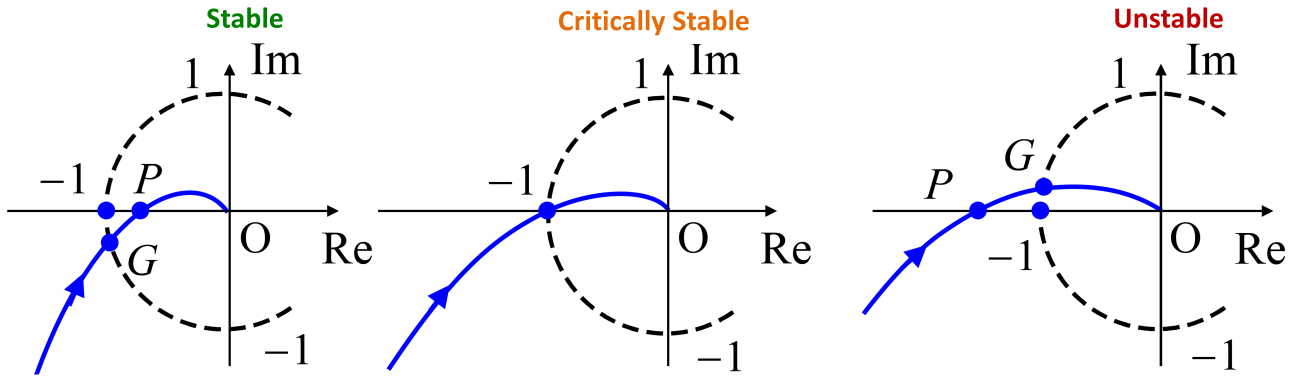
Stability: Nyquist Example

$$L(s) = \frac{K}{s(s+1)(s+2)} \qquad K = 3,\ 6,\ 12$$

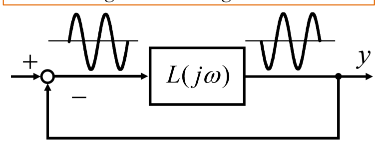




Continue



Stable	Critically Stable	Unstable
$\angle P(j\omega_{pc})K(j\omega_{pc}) = -180^\circ$	$\angle P(j\omega_{pc})K(j\omega_{pc}) = -180^\circ$	$\angle P(j\omega_{pc})K(j\omega_{pc}) = -180^\circ$
$\lvert P(j\omega_{pc})K(j\omega_{pc})\rvert < 1$	$\lvert P(j\omega_{pc})K(j\omega_{pc})\rvert = 1$	$\lvert P(j\omega_{pc})K(j\omega_{pc})\rvert > 1$
$\angle P(j\omega_{gc})K(j\omega_{gc}) > -180^\circ$	$\angle P(j\omega_{gc})K(j\omega_{gc}) = -180^\circ$	$\angle P(j\omega_{gc})K(j\omega_{gc}) < -180^\circ$



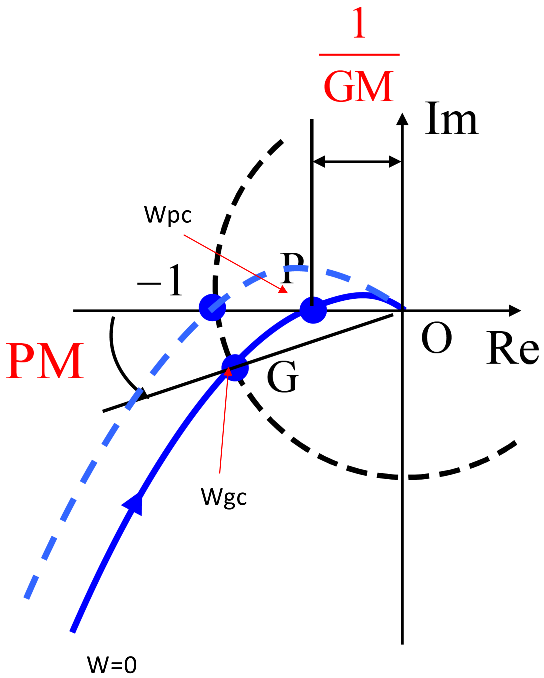
Nyquist: Phase and Gain Margins

$$GM = \frac{1}{OP} \quad (dB)$$

$$PM = \angle GOP \quad (^{\circ})$$

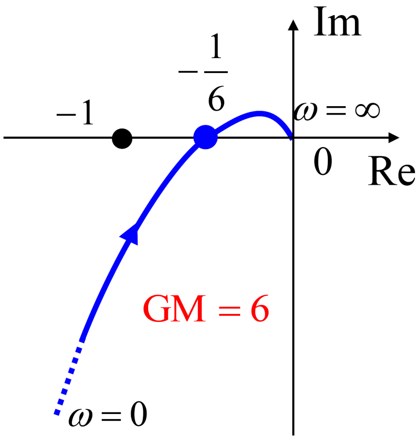
margin

```
[Gm,Pm,Wcg,Wcp] = margin(SYS)
```



Nyquist

$$L(s) = \frac{1}{s(s+1)(s+2)}$$



Stable

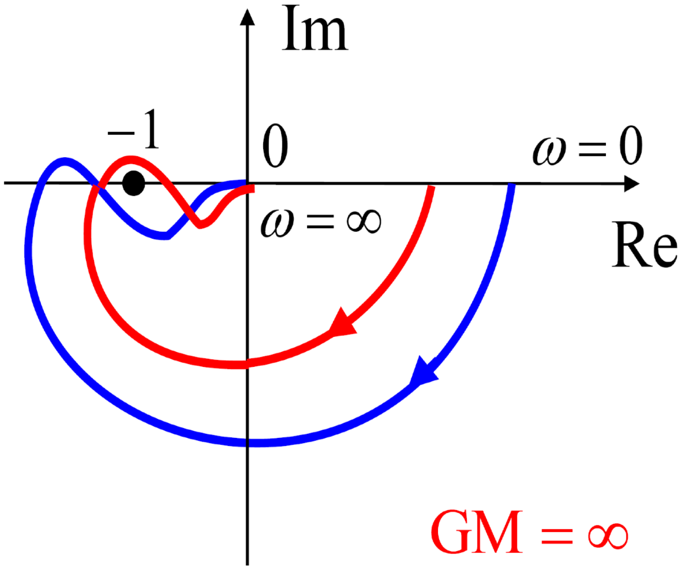
margin

```
num=[0 0 0 1];
den=conv(conv([1,0],[1,1]),[0,1,2]);
L=tf(num,den);
[Gm,Pm,Wcg,Wcp] = margin(L)
```

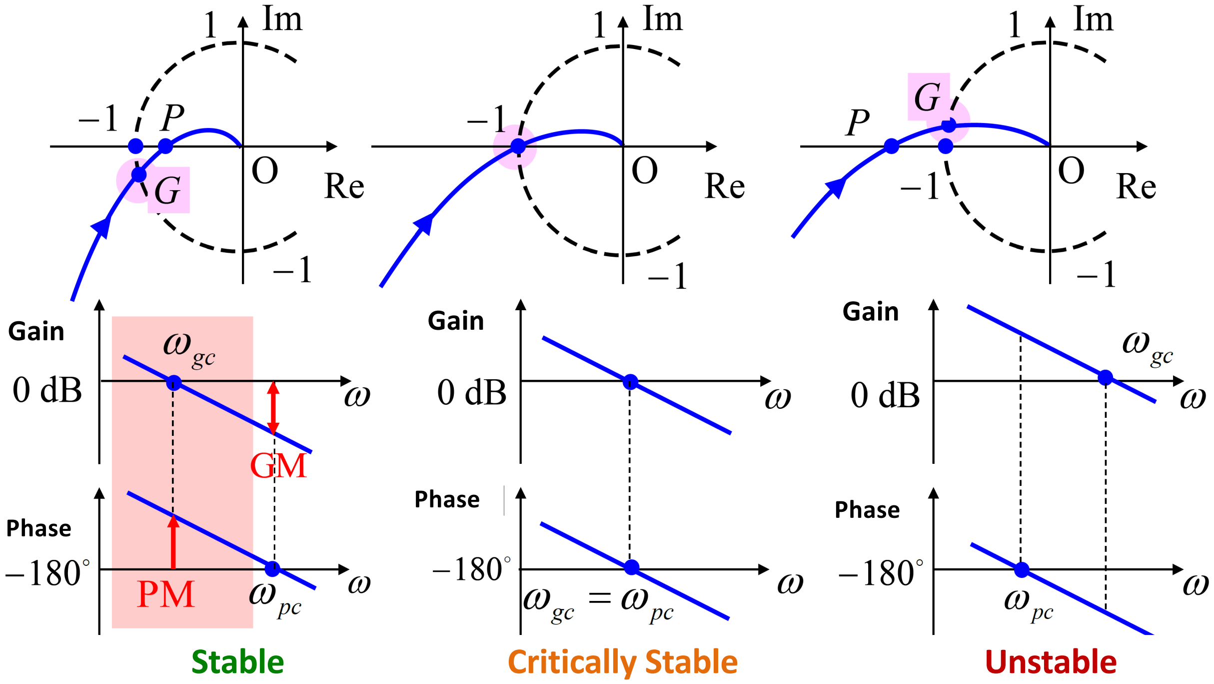
```
>>
Gm = 6.0000

Pm = 53.4109

Wcg =  1.4142

Wcp =  0.4457
```

Nyquist

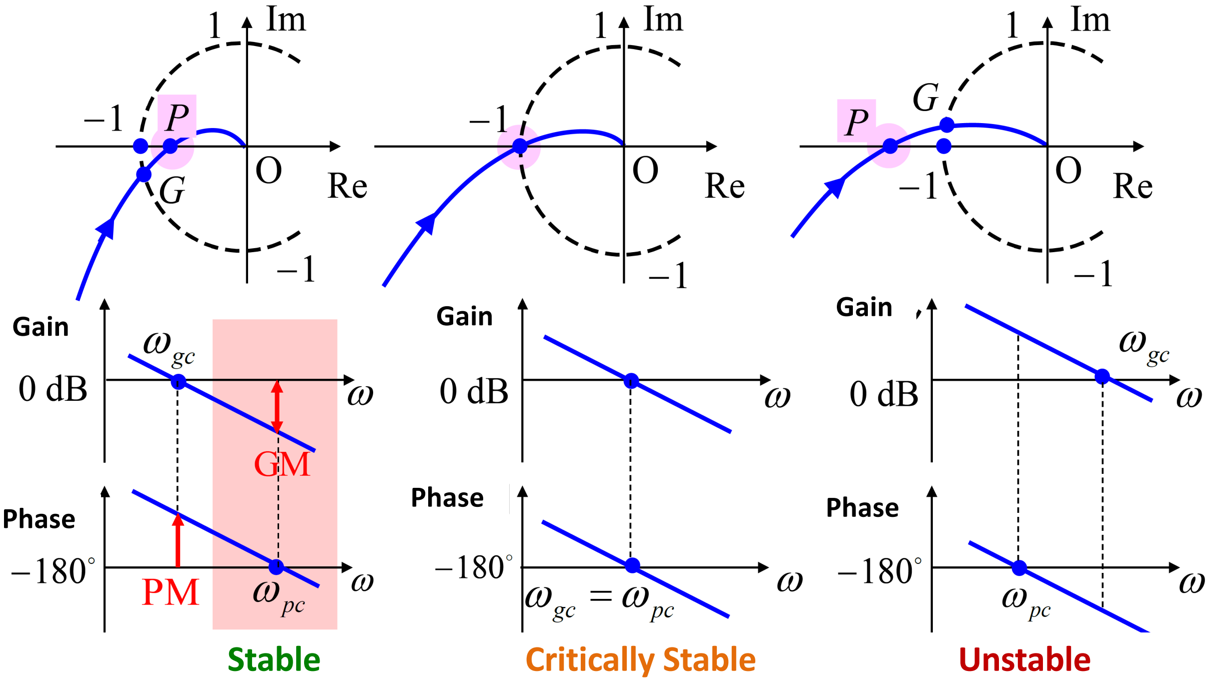
Conditionally Stable



Nyquist and Bode



Nyquist and Bode



Bode: Phase and Gain Margins

$$L(s) = \frac{K}{s(s+1)(s+2)} \qquad (K = 3)$$

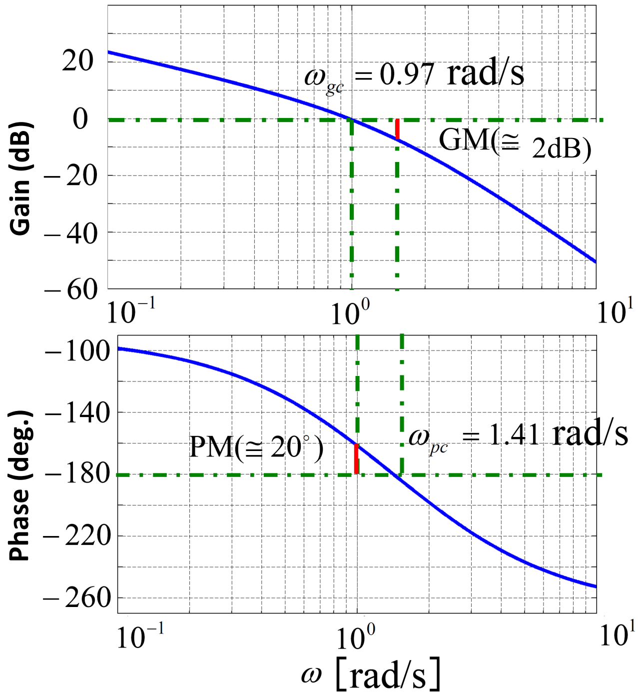
$$\omega_{gc} \cong 0.97 \text{ rad/s}$$

$$\text{PM} \cong 20^\circ$$

$$\omega_{pc} \cong 1.41 \text{ rad/s}$$

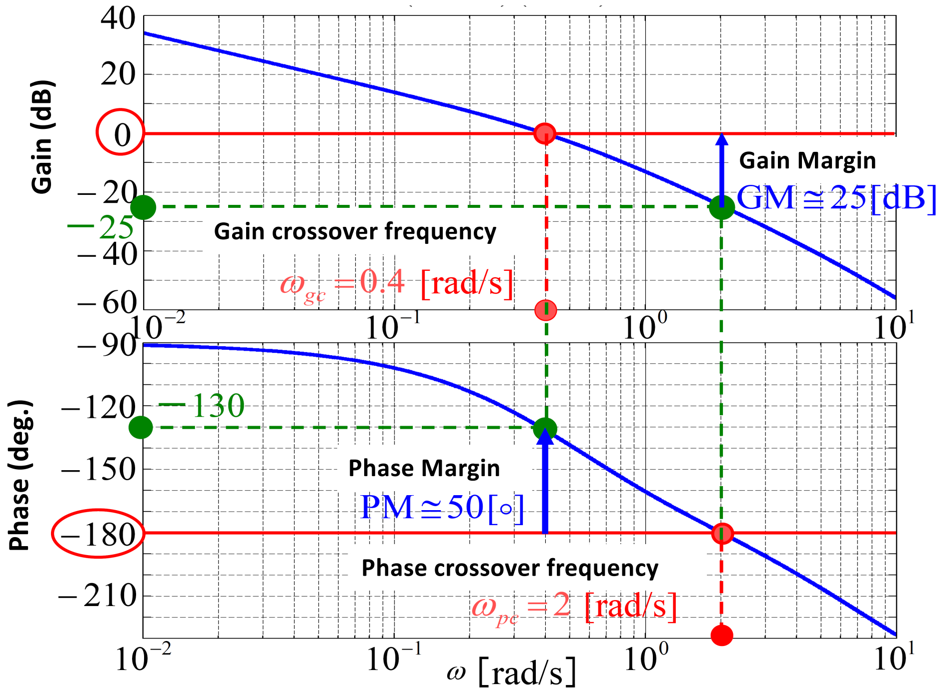
```
num=[0 0 0 3];
den=conv(conv([1,0],[1,1]),[0,1,2]);
L=tf(num,den);
bode(L)
[Gm,Pm,Wcg,Wcp] = margin(L)
```

```
Gm =2.0000
Pm =20.0381
Wcg =1.4142
Wcp =0.9693
```



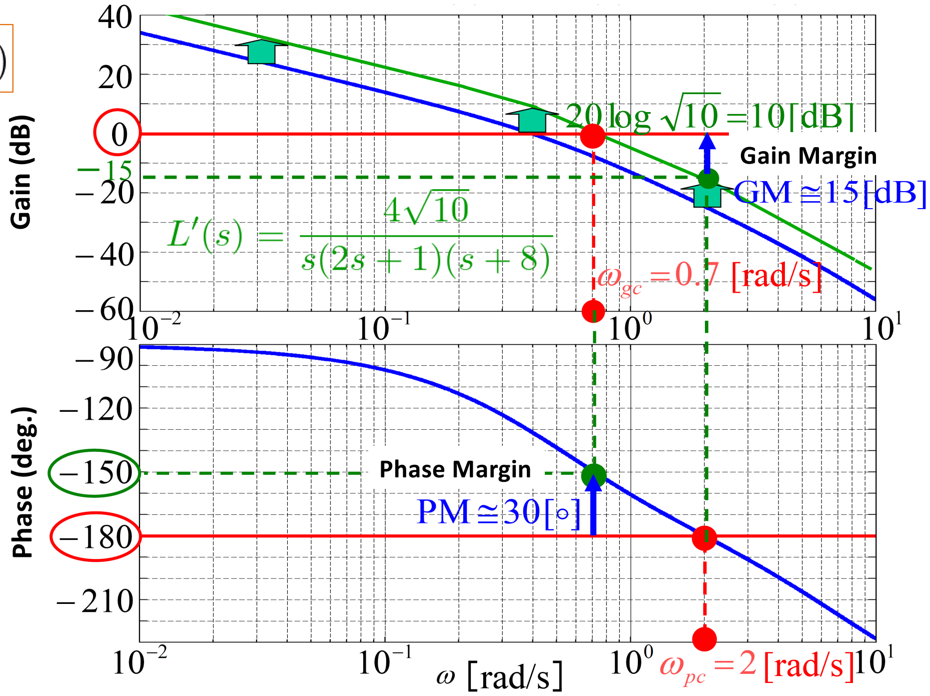
Bode: Phase and Gain Margins

$$L(s) = \frac{4}{s(2s+1)(s+8)}$$



Bode: Phase and Gain Margins

$$L'(s) = KL(s) = \sqrt{10}L(s)$$



Project: Report 1

Using MATLAB for your selected system:

1) **Plot the step response,**
2) **Plot root locus, and discuss the system stability,**
3) **Plot Bode diagram,**
4) **Find phase margin and gain margin,**
5) **Plot Nyquist diagram.**

Deadline: The day before next Meeting

Please only use this email address: bevranih18@gmail.com

Thank You!