Robust Control Systems

Loop Shaping Control Design

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Contents

Reference

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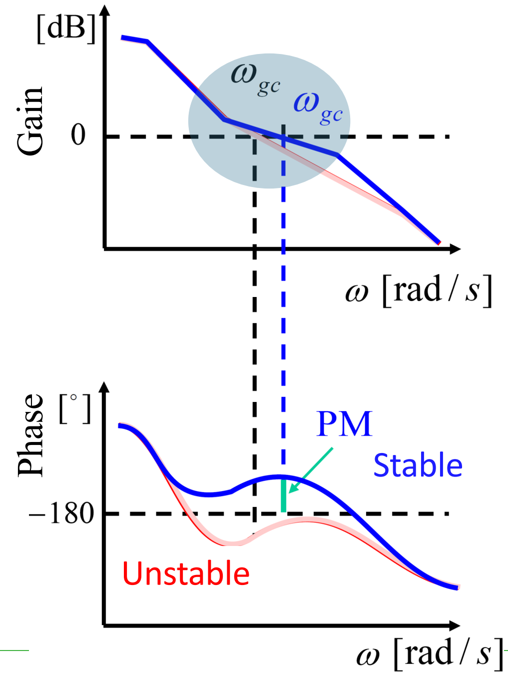
2. M. Fujita, **Lecture Notes on Feedback Control Systems**, Tokyo Institute of Technology, 2019.

3. R. Smith, **Lecture Notes on Control Systems**, ETH Zurich, 2020.

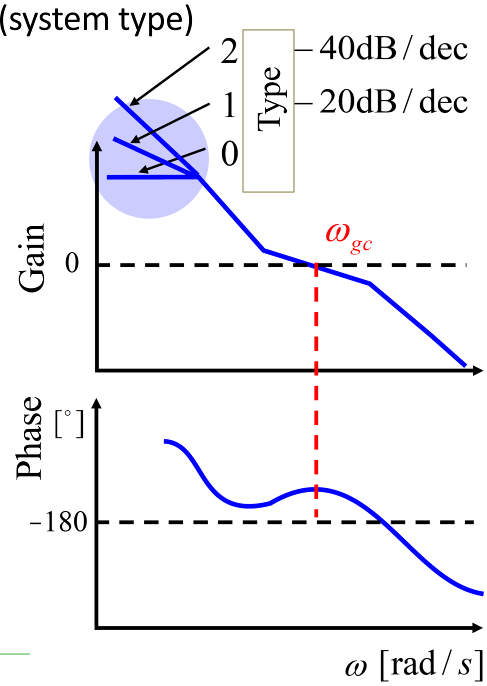
Key points for Loop Shaping

❑**Improve Stability Margin**



❑**Improve low frequency behavior**

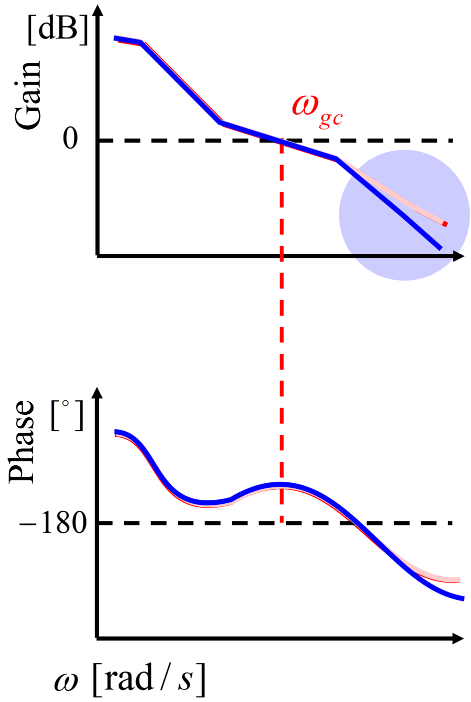
(system type)



Key points for Loop Shaping

- **Improve high frequency behavior**

  (Noise reduction)



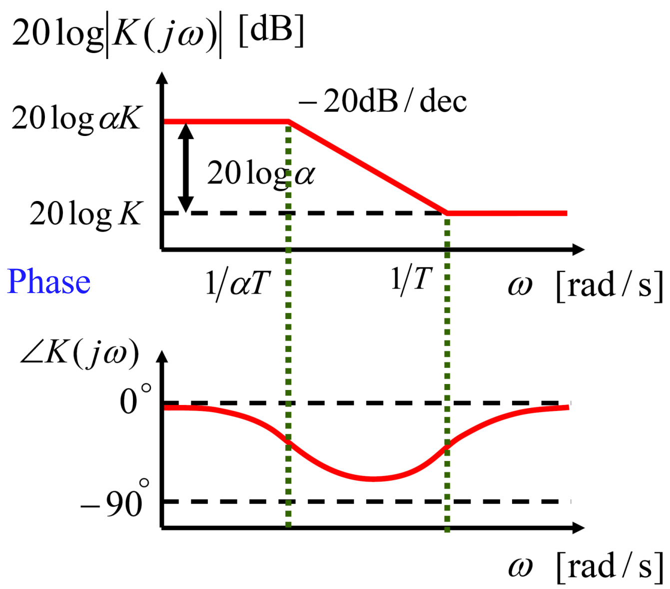
Phase Lead-lag Compensator Design

$$K(s) = K\frac{\alpha(Ts+1)}{\alpha Ts+1}$$

Improvement of steady-state characteristics

$$+20\log\alpha[\text{dB}]$$
$$\left(K(0)=\alpha K, K(\infty)=K\right)$$

The corner frequency (T) must be adjusted, appropriately



Phase Lag Compensator Design

❑Comparison with PI

$$K(s) = K\frac{\alpha(Ts+1)}{\alpha Ts+1} \quad (\alpha > 1)$$

$+20\log\alpha$ [dB]

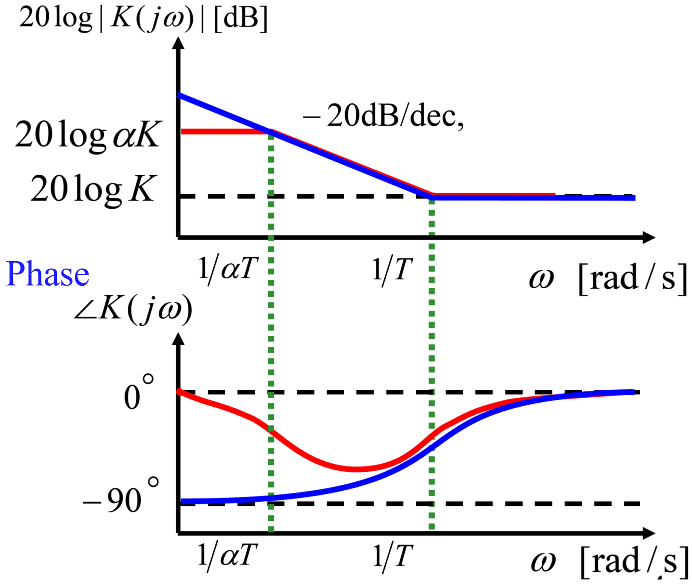
$\left[K(0) = \alpha K,\ K(\infty) = K\right]$

$\alpha \to \infty$

$$K(s) = K\left(1+\frac{1}{Ts}\right)$$

$$K_{PI}(s) = K_P\left(1+\frac{1}{T_I s}\right)$$

$T_I = \dfrac{K_P}{K_I}$ (Integration time)



$\left[K(0) \approx \infty,\ K(\infty) = K_P\right]$

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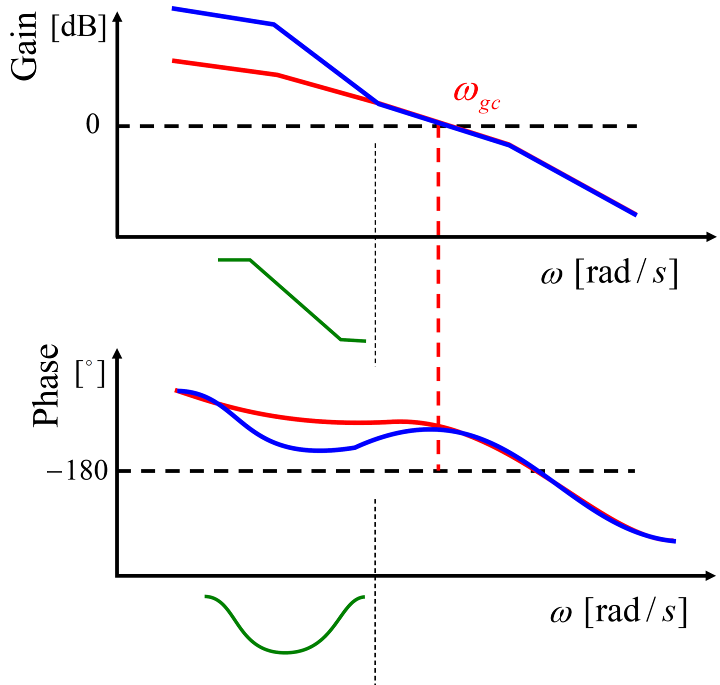


A Design Example

$$P(s) = \frac{10}{s(s+1)(s+10)}$$

Find Controller K(s) to satisfy: $K_v \geq 10$, $\mathrm{PM} \geq 40°$

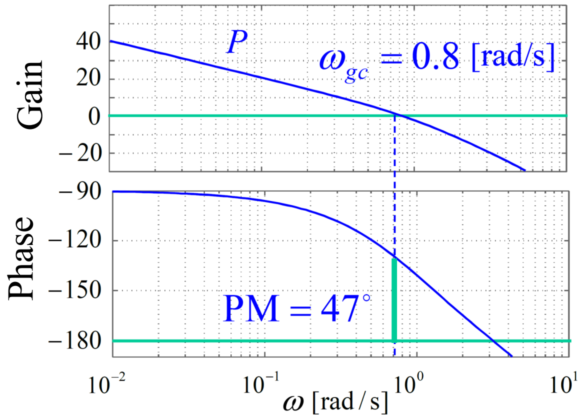
Step 1: Focusing on the PM and gain crossover frequency, the controller **gain K** must be determined so that the desired transient response characteristics can be obtained.

Gain crossover frequency: $\omega_{gc} \approx 0.8$

Phase margin: $\mathrm{PM} = 47°$

which meets $\mathrm{PM} \geq 40°$

$\Rightarrow K = 1$



A Design Example

Step 2: Draw bode diagram of open-loop transfer function resulted from Step 1 and evaluate its low frequency gain.

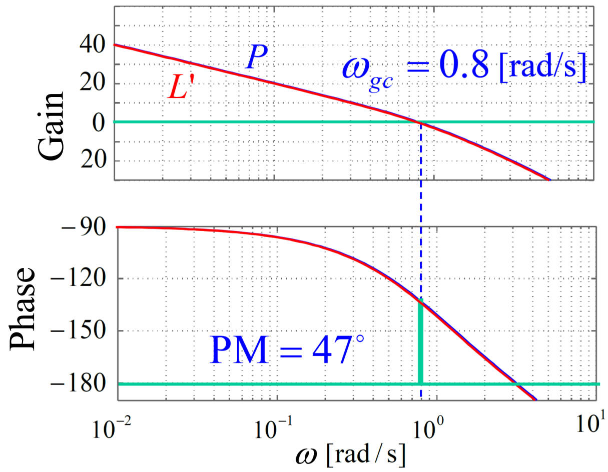
Open-loop transfer function:

$$L' = PK = \frac{10}{s(s+1)(s+10)} \quad (K=1)$$

Speed deviation constant:

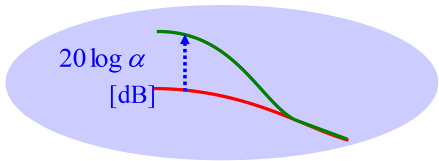
$$K_v' = \lim_{s \to 0} sL'(s) = \lim_{s \to 0} \frac{10}{(s+1)(s+10)} = 1$$

The required low frequency gain is 10 times or more: $K_v \geq 10$



A Design Example

Step 3: Considering that the low-frequency gain increases by $+20\log\alpha[\mathrm{dB}]$ the parameter α must be determined to satisfy the required steady-state characteristics.

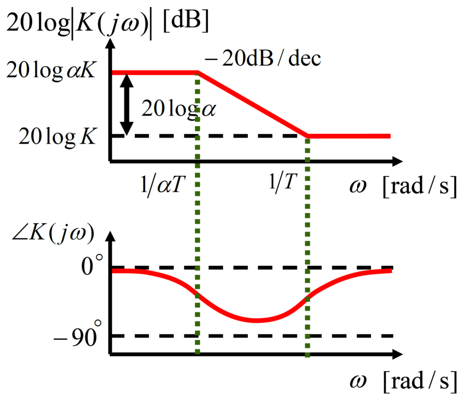


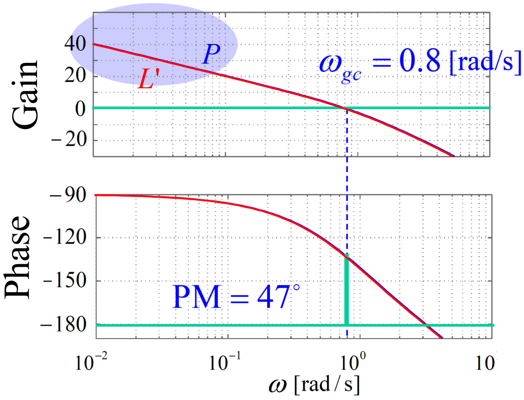
The minimum required low frequency gain is :

$$K_v \geq 10$$

⇩

$$\alpha = 10$$



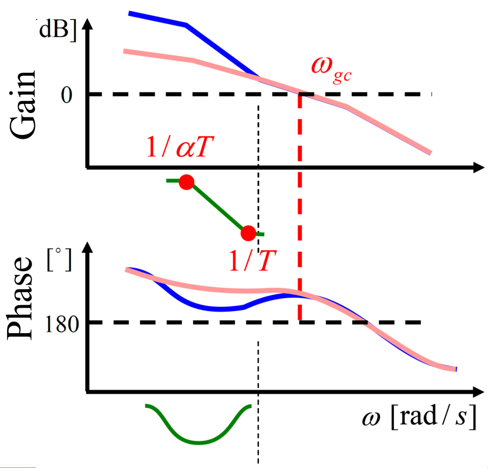


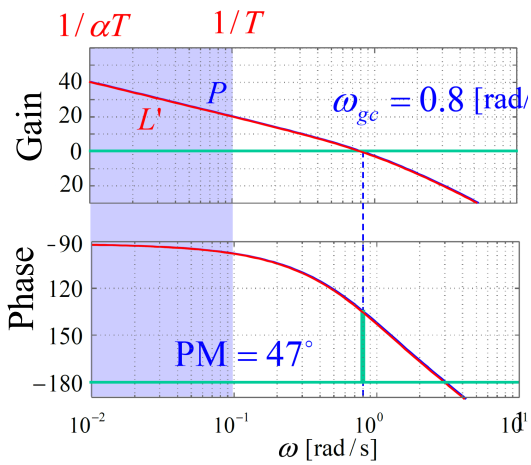
A Design Example

Step 4: To keep the stability against the phase delay, select the corner frequency $\omega = 1/T$ about 1 dec below the gain crossover frequency. Then determine second corner frequency $\omega = 1/(\alpha T)$.

$T = 10$ $(\omega{=}0.1)$: the corner frequency is sufficiently smaller than the gain crossover frequency

$$\frac{1}{\alpha T} = 0.01, \quad \frac{1}{T} = 0.1$$





A Design Example

Step 5: Construct the compensator using the determined parameters (K, α, T).

$L(s) = P(s)K(s)$

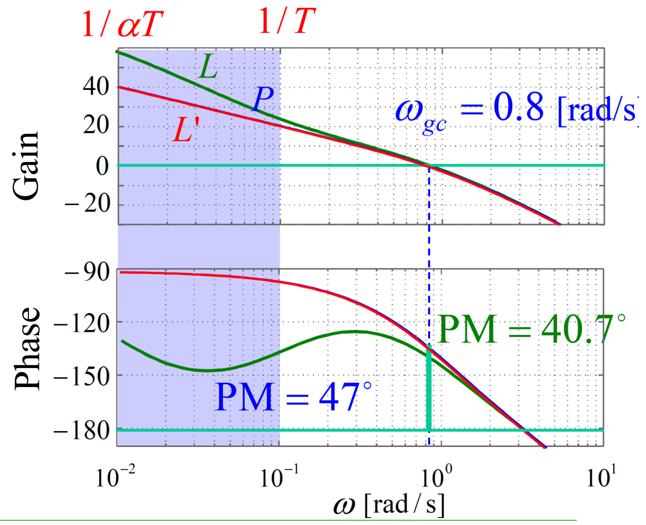
Phase lag compensator: $K(s) = K\dfrac{\alpha(Ts+1)}{\alpha Ts+1}$

$K = 1, \quad \alpha = 10, \quad T = 10$

$$K(s) = 1\cdot\frac{10(10s+1)}{10\cdot 10s+1} = \frac{s+0.1}{s+0.01}$$

Gain crossover frequency: $\omega_{gc} = 0.8$ rad/s

Phase margin: $\mathrm{PM} \geq 40^\circ$



A Design Example

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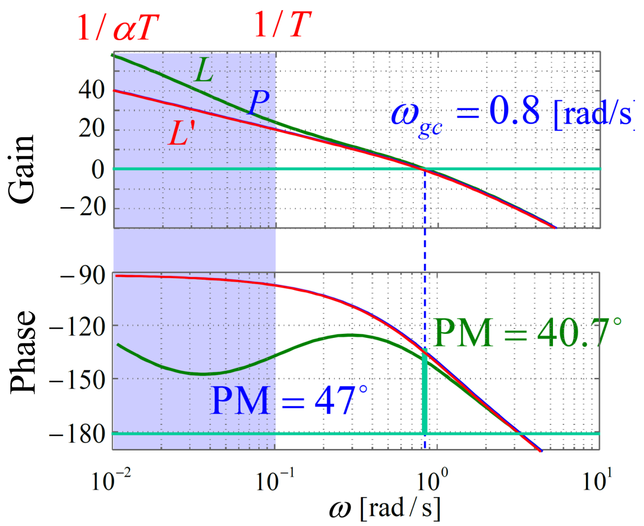


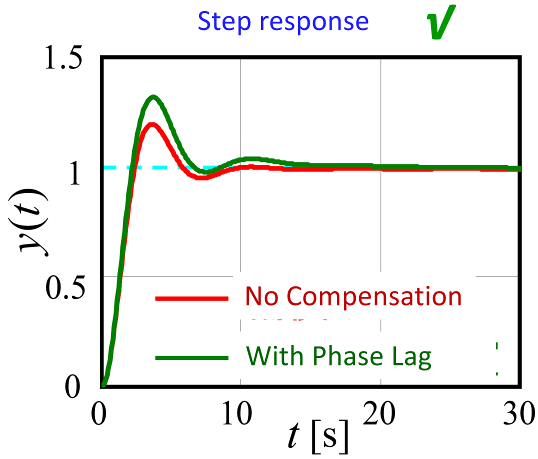
Evaluation

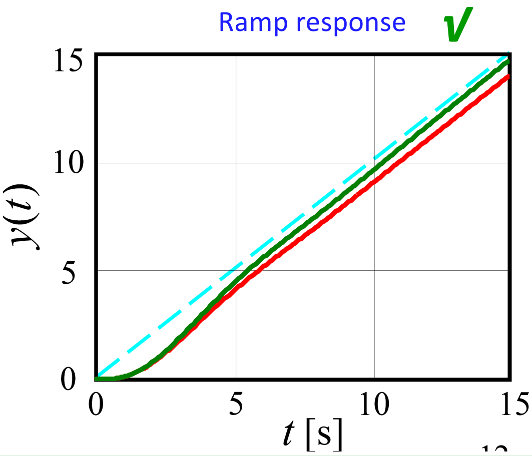
Speed deviation constant (steady characteristic): $K_v \geq 10$ √

Phase margin (damping characteristic): PM ≥ 40 √

$$L(s) = P(s)K(s) = \frac{10(s+0.1)}{s(s+0.01)(s+1)(s+10)}$$

$$K_v = \lim_{s\to 0} sL(s) = \frac{1}{0.1} = 10$$ √ PM $\geq 40°$ $\omega_{gc} \approx 0.8$ [rad/s] √





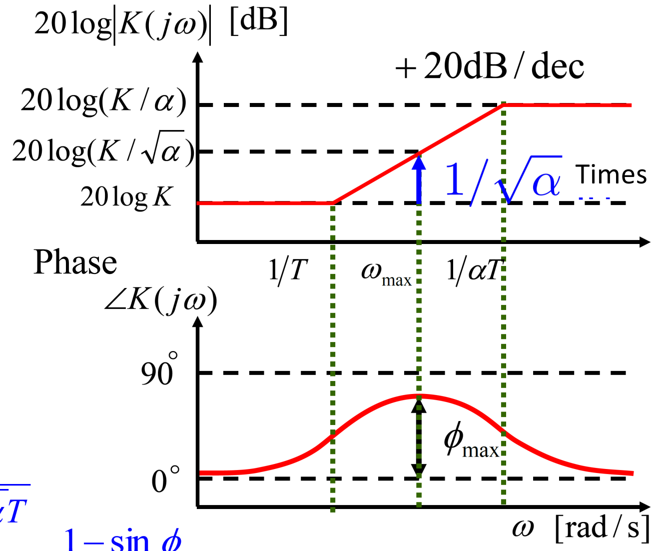
Phase Lead Compensator Design

$$K(s) = K\frac{Ts+1}{\alpha Ts+1} \quad (\alpha < 1)$$

Stabilizing transient characteristics

Phase lead $\frac{1}{T} < \omega < \frac{1}{\alpha T}$

Note: High frequency gain is increased. Noise amplification degrades Robust stability!



Angular frequency where the phase advances most

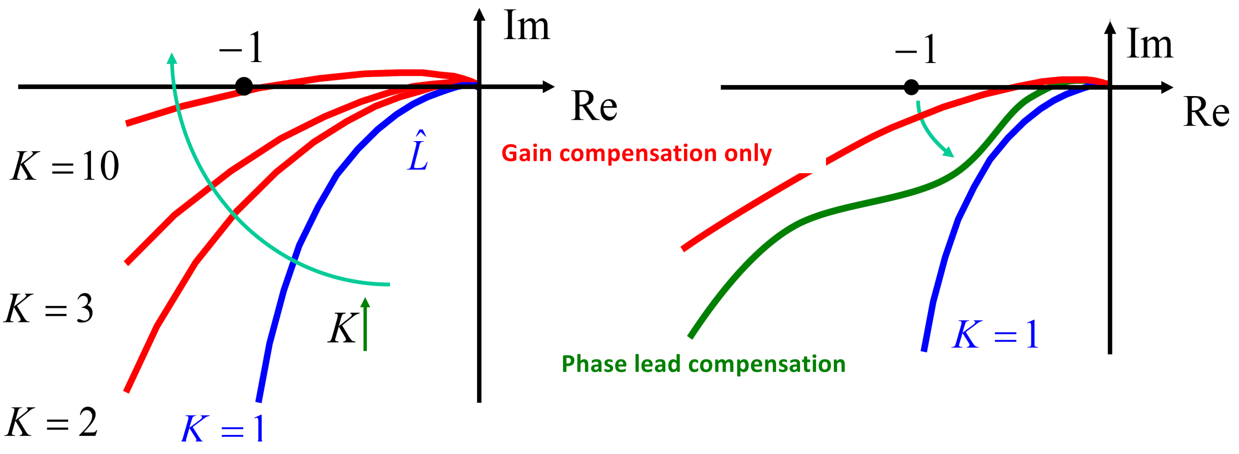
$$\omega_{max} = \frac{1}{\sqrt{\alpha}T}$$

Maximum value of phase lead: $\sin\phi_{max} = \frac{1-\alpha}{1+\alpha} \Rightarrow \alpha = \frac{1-\sin\phi_{max}}{1+\sin\phi_{max}}$

$\omega = \omega_{max}$, the gain becomes $\frac{1}{\sqrt{\alpha}}$ times.

Phase Lead Compensator Design

$\hat{L} = PK,$



Phase Lead Compensator Design

❑ Comparison with PD

$$K(s) = K\frac{Ts+1}{\alpha Ts+1} \quad (\alpha < 1)$$

$+\phi_{max}[°]$

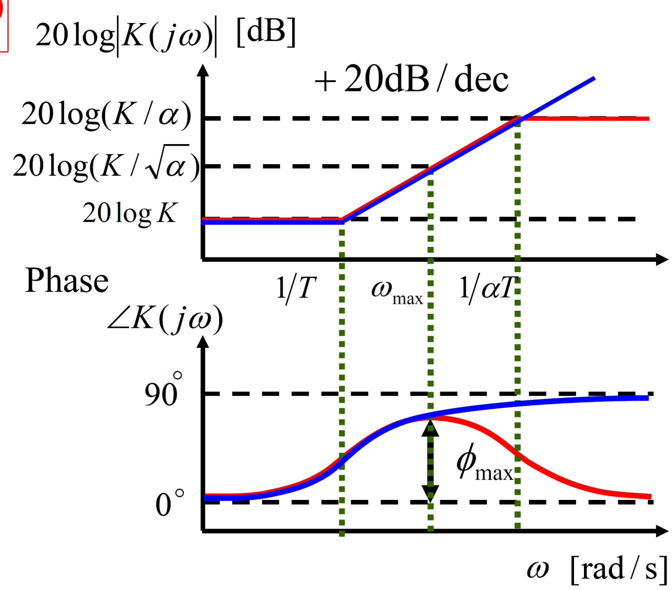
$$\left(K(0) = K,\ K(\infty) = K/\alpha \right)$$

$\alpha \to 0$

$$K(s) = K(Ts+1)$$

$$K_{PD}(s) = K_P(T_D s+1)$$

Note: It is difficult to realize an ideal differentiator



$$K'_{PD} = \frac{K_P(1+T_D s)}{1+(T_D/N)s} \quad (3 \le N \le 20)$$

Phase lead compensator

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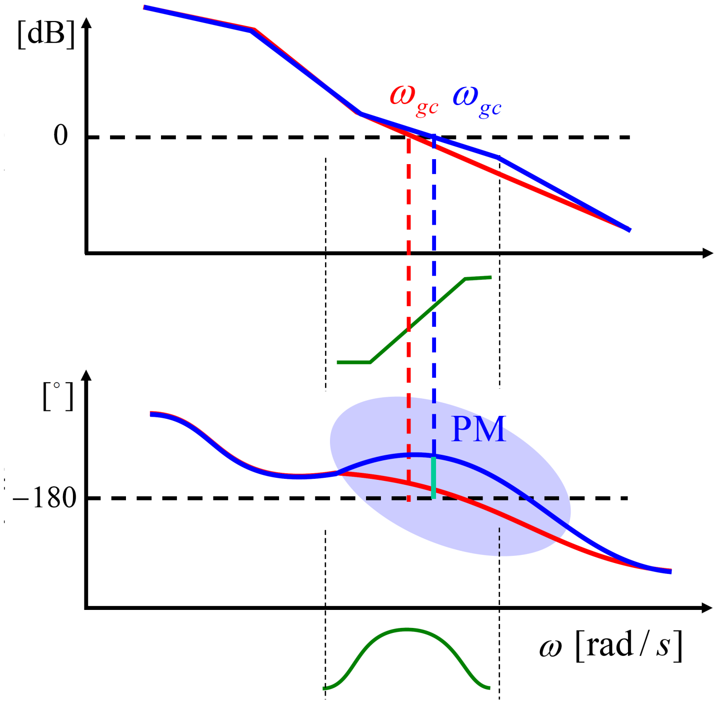


Phase-lead Compensator: Design Example

$$P(s)=\frac{10}{s(s+1)(s+10)}$$

Find Controller K(s) to satisfy: $\mathrm{PM}\approx 40^\circ$ $\omega_{gc}\geq 2$ [rad/s]

Step 1: To meet the specifications for quick response and steady-state characteristics, determine the value of gain compensation K.

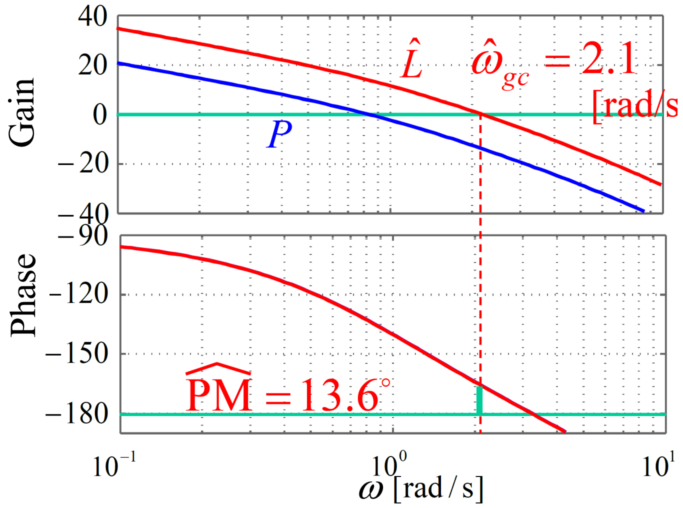
Gain compensation: $K=5$

Open-loop transfer function:

$$\hat{L}(s)=\frac{50}{s(s+1)(s+10)}$$

Gain crossover frequency:

$\hat{\omega}_{gc}=2.1>2$ [rad/s] ⇨ $\omega_{gc}\geq 2$ [rad/s] ✔



Phase-lead Compensator: Design Example

Step 2: Draw a Bode plot of the open-loop transfer function $\hat{L}(s) = KP(s)$ using K in *Step 1* and evaluate its phase margin $\widehat{\mathrm{PM}}$. The required phase lead amount is the difference between the given phase margin PM and this one ($\hat{\phi} = \mathrm{PM} - \widehat{\mathrm{PM}}$). Considering an appropriate margin (e.g, 5 deg. or more), set it as $\phi_{\max} = \hat{\phi} + 5°$ or more.

Phase margin: $\widehat{\mathrm{PM}} = 13.6^{\circ}$

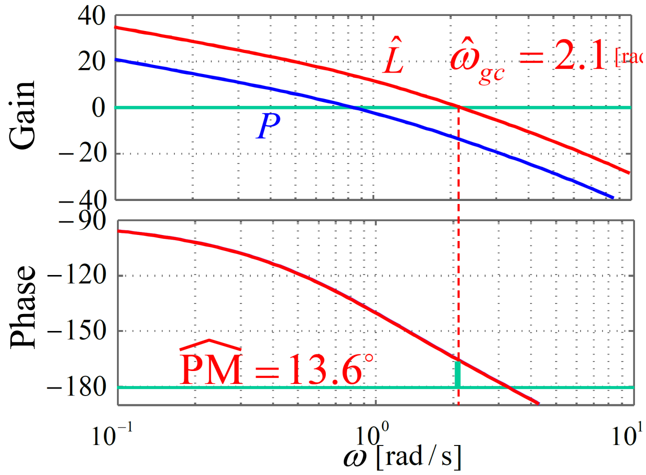
Required phase margin: $\mathrm{PM} \approx 40°$

$$\hat{\phi} = \mathrm{PM} - \widehat{\mathrm{PM}} = 40 - 13.6 = 26.4°$$

(Required phase lead amount)

$$\phi_{\max} = \hat{\phi} + 10° = 36.4$$

(Margin)



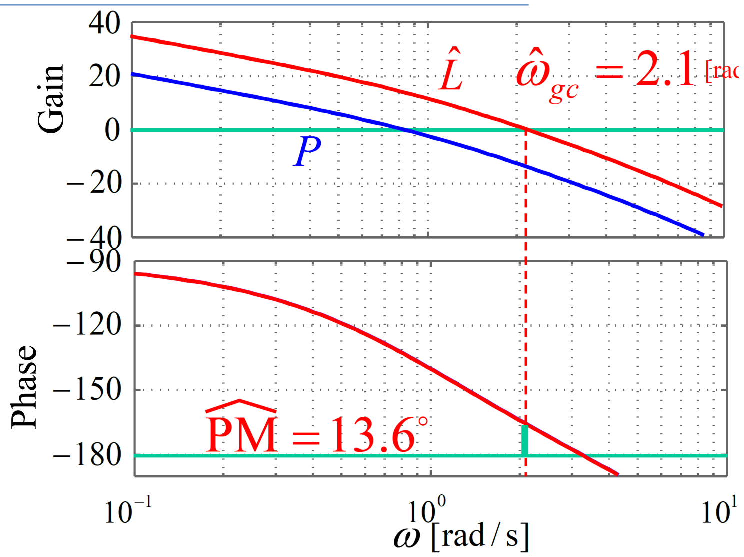
Phase-lead Compensator: Design Example

Step 3: Determine the value of α from $\alpha = \dfrac{1 - \sin \phi_{\max}}{1 + \sin \phi_{\max}}$.

$$K(s) = K \frac{Ts + 1}{\alpha Ts + 1}$$

$$\alpha = \frac{1 - \sin \phi_{\max}}{1 + \sin \phi_{\max}}$$

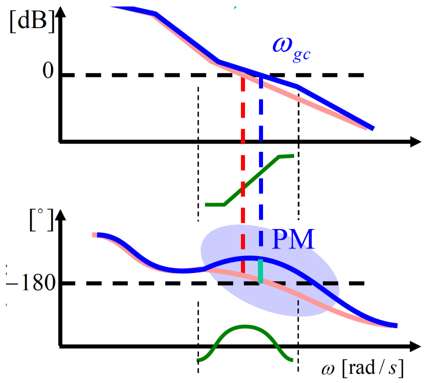
$$\phi_{\max} = 36.4° \Rightarrow \alpha = 0.255$$

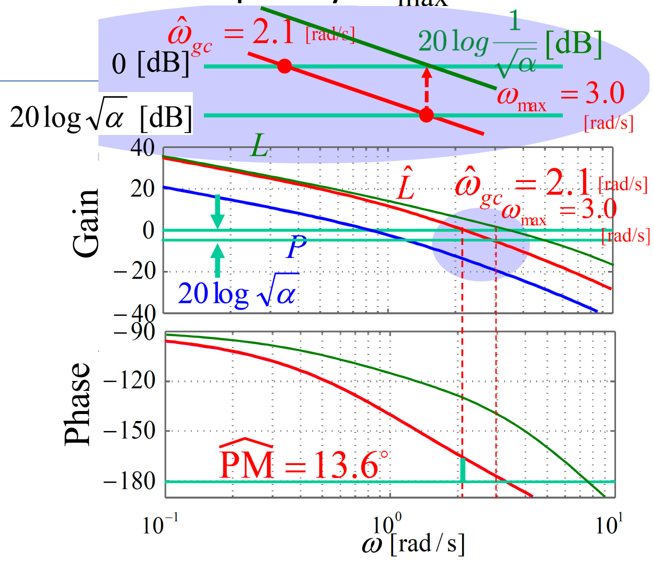


Phase-lead Compensator: Design Example

Step 4: In phase lead compensation, the gain increases by $1/\sqrt{\alpha}$ at the angular frequency where the phase leads, so the angular frequency where $|\hat{L}(j\omega)|$ is $\sqrt{\alpha}$ ($= 20\log\sqrt{\alpha}$ [dB]) is set as the new gain crossover frequency $\omega_{\max}$ after compensation.

$$|\hat{L}(j\omega_{\max})| = \sqrt{\alpha} = 0.505 \Rightarrow \omega_{\max} = 3.0 \text{ [rad/s]}$$





Phase-lead Compensator: Design Example

Step 5: The value of parameter T is determined from $\omega_{\max} = \frac{1}{\sqrt{\alpha}T}$. At this time, the corner frequency of phase lead compensation is

$$1/T = \omega_{\max}\sqrt{\alpha}, \quad 1/(\alpha T) = \omega_{\max}/\sqrt{\alpha}$$

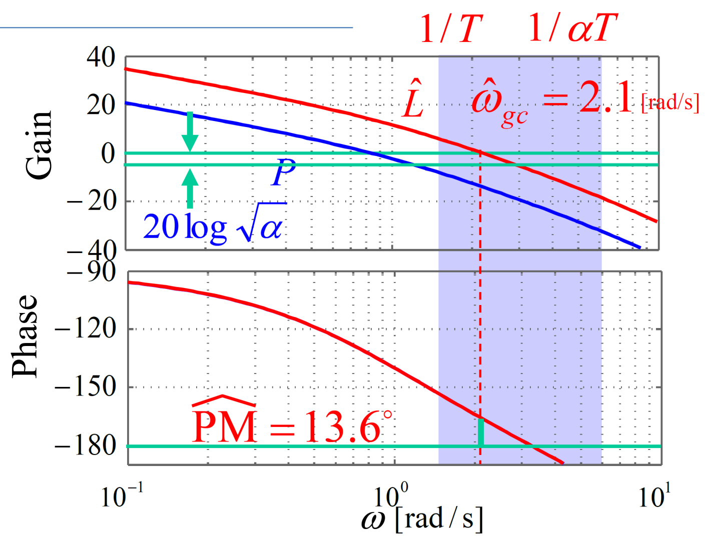
$$\omega_{\max} = \frac{1}{\sqrt{\alpha}T} \Rightarrow T = \frac{1}{\sqrt{\alpha}\omega_{\max}}$$

$\omega_{\max} = 3.0$ [rad/s], $\alpha = 0.255$

$\Rightarrow T = 0.660$

Frequency corners:

$$\frac{1}{T} = 1.52, \qquad \frac{1}{\alpha T} = 5.94$$



Phase-lead Compensator: Design Example

Step 6: After finding design parameters K, α, T, the phase lead controller is constructed as follows: $K(s) = K\dfrac{Ts+1}{\alpha Ts+1}$

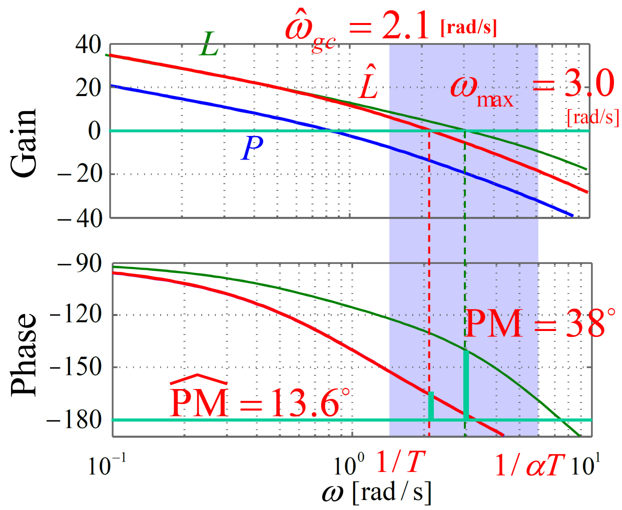
$$K(s) = K\frac{Ts+1}{\alpha Ts+1}$$

$K = 5, \alpha = 0.255, \quad T = 0.660$

$$K(s) = 5 \cdot \frac{0.66s+1}{0.255 \cdot 0.66s+1}$$
$$= \frac{19.6(s+1.52)}{s+5.94}$$

Gain crossover frequency $\omega_{gc} = 3.0$ [rad/s]

Phase margin $\mathrm{PM} = 38^\circ$



Evaluation

Gain crossover frequency (rapid response):

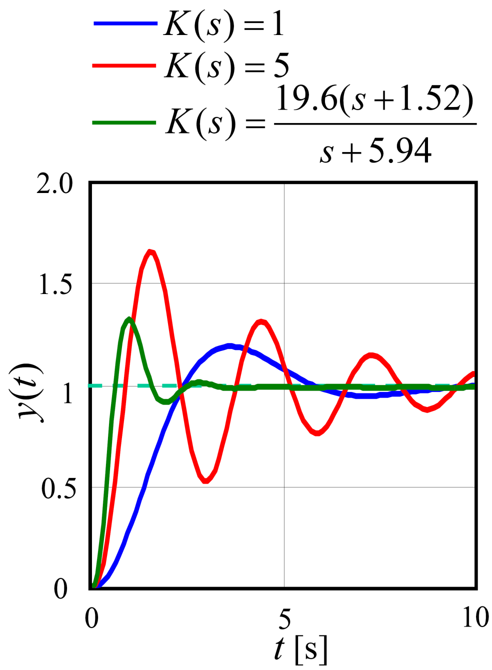
$$\omega_{gc} \geq 2 \ \text{[rad/s]}$$

Phase margin (damping characteristic): $\mathrm{PM} \approx 40^\circ$

$\omega_{gc} = 3.0 (= \omega_{max})$ [rad/s] ✓

$\mathrm{PM} \cong 38^\circ$ ✓

Step response ✓



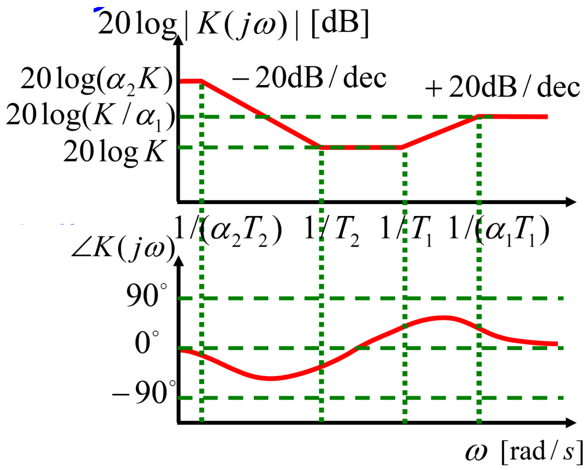
Phase Lead-lag Compensator Design

$$K(s) = K\underbrace{\left(\frac{T_1 s+1}{\alpha_1 T_1 s+1}\right)}_{\text{Lead}}\underbrace{\left(\frac{\alpha_2 (T_2 s+1)}{\alpha_2 T_2 s+1}\right)}_{\text{Lag}} \quad (\alpha_1 < 1, \alpha_2 > 1)$$

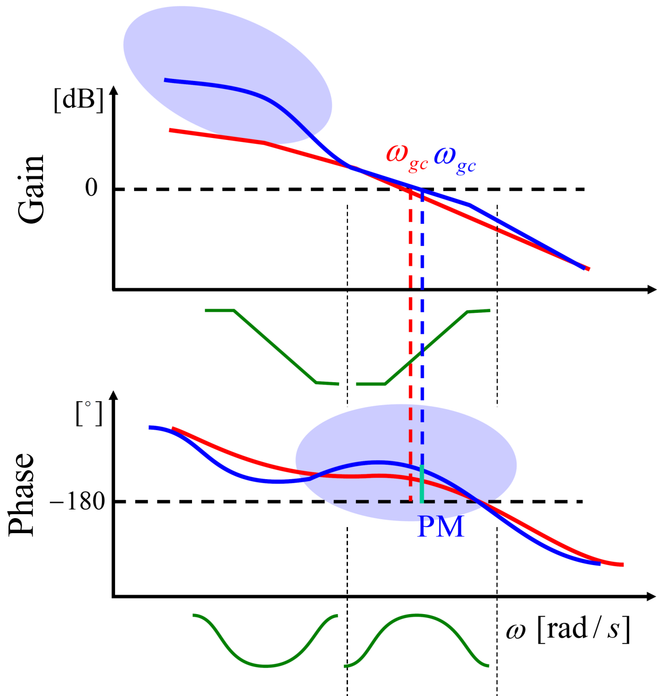
We may need multiple stages to improve of steady-state characteristics and transient characteristics

$$+20\log \alpha_2 \ [\text{dB}]$$

Phase lead: $\frac{1}{T_1} < \omega < \frac{1}{\alpha_1 T_1}$



Phase Lead-lag Compensator Design



Phase Lead-lag Compensator Design

❑ Comparison with PID

$$K(s) = K\frac{T_1 s+1}{\alpha_1 T_1 s+1}\,\frac{\alpha_2 (T_2 s+1)}{\alpha_2 T_2 s+1} \quad (\alpha_1 < 1,\ \alpha_2 > 1)$$

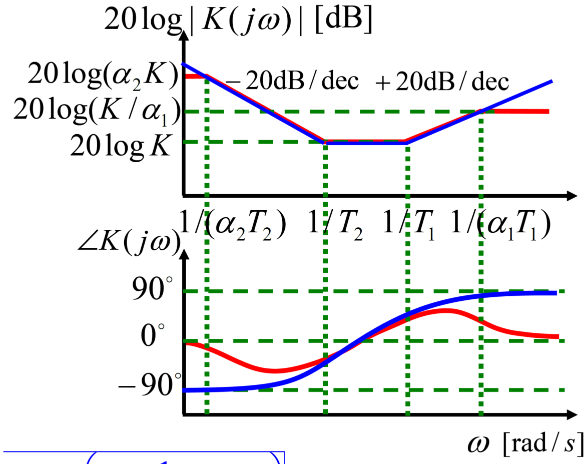
$+20\log\alpha$ [dB], $+\phi_{\max}$ [°]

$\left[K(0) = \alpha_2 K,\ K(\infty) = K/\alpha_1\right]$

$$\alpha_1 \to 0,\ \alpha_2 \to \infty$$

$$K(s) = K\underbrace{(T_1 s+1)}_{\text{PD}}\underbrace{\left(1+\frac{1}{T_2 s}\right)}_{\text{PI}}$$

$$= \frac{K(T_1+T_2)}{T_2}\left(1+\frac{1}{(T_1+T_2)s}+\frac{T_1T_2}{T_1+T_2}s\right) \Rightarrow K_{PID}(s) = K_P\left(1+\frac{1}{T_I s}+T_D s\right)$$



Example

$$P(s) = \frac{0.01}{s^2+0.04s+0.01} \quad \xrightarrow{\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}} \quad \omega_n = 0.1\text{rad/s},\ \zeta = 0.2$$

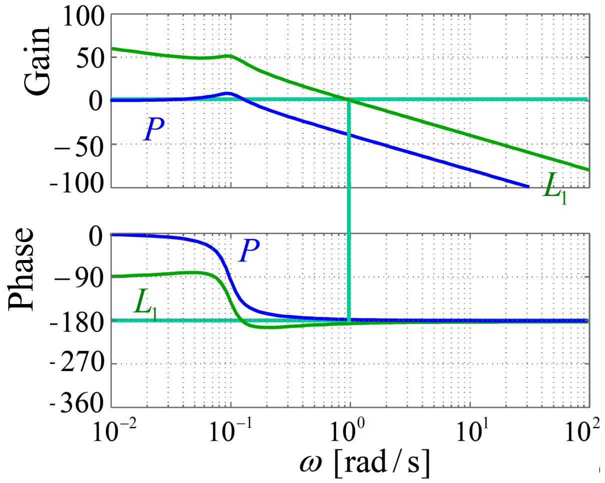
Phase lag (PI controller):

$$K_{PI}(s) = 100\frac{s+0.1}{s}$$

$$L_1 = PK_{PI}$$

⇨ Low frequency gain: large

Phase margin: PM= -3.37deg



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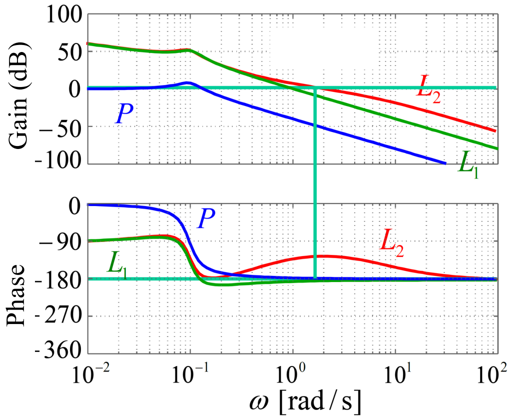
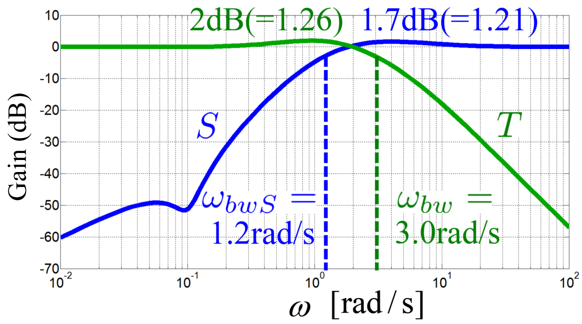
Phase lead: $K_L(s) = \frac{14.3(s+0.53)}{s+7.52}$ $K=1,\ \alpha = 0.07,\ T = 1.9$

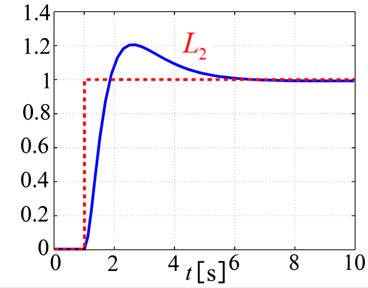
Lead-lag compensator: $K_{LL}(s) = \frac{1430(s+0.1)(s+0.53)}{s(s+7.52)}$

$L_2 = PK_{LL}$

PM= 58.5deg $\omega_{gc} = 1.92$rad/s







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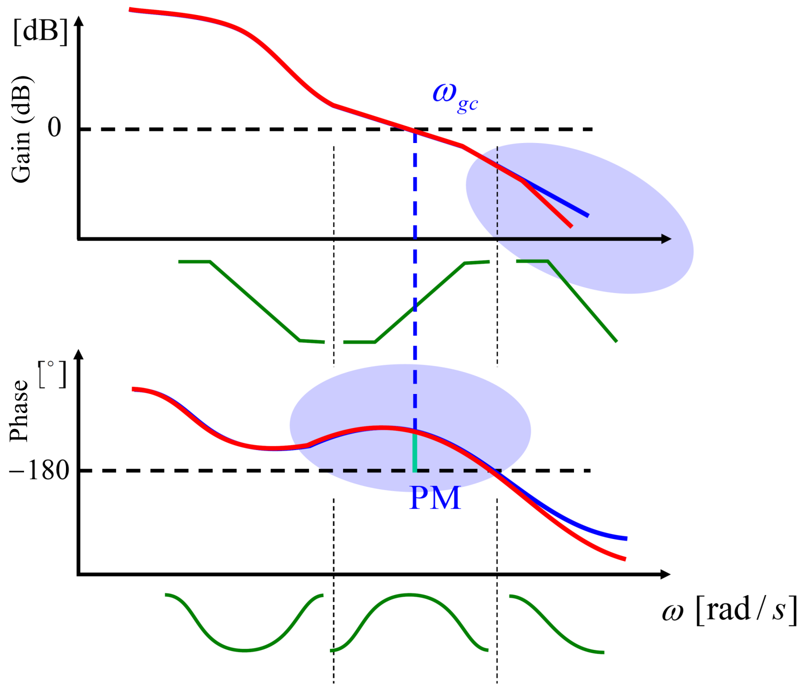
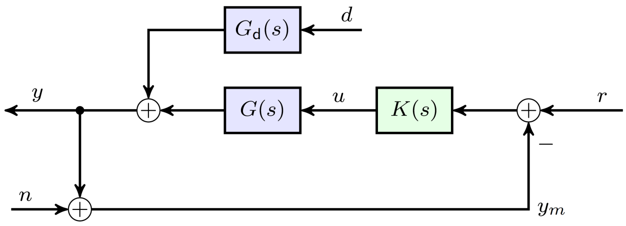


Loop Shaping Control



$$e = \underbrace{\frac{1}{1+L(s)}}_{S(s)} r - \underbrace{\frac{1}{1+L(s)}}_{S(s)} G_d(s) d + \underbrace{\frac{L(s)}{1+L(s)}}_{T(s)} n$$

Performance requirements can be approximated by requirements for $L(j\omega)$.

$|L(j\omega)| \gg 1 \implies |S(j\omega)| \ll 1$ (good tracking performance)

$|L(j\omega_c)| = 1$ gives bandwidth $\approx \omega_c$

$|L(j\omega)| \ll 1 \implies |T(j\omega)| \ll 1$ (good noise rejection)

Inverse-based Control Design

Stable, minimum-phase plant

Can choose:

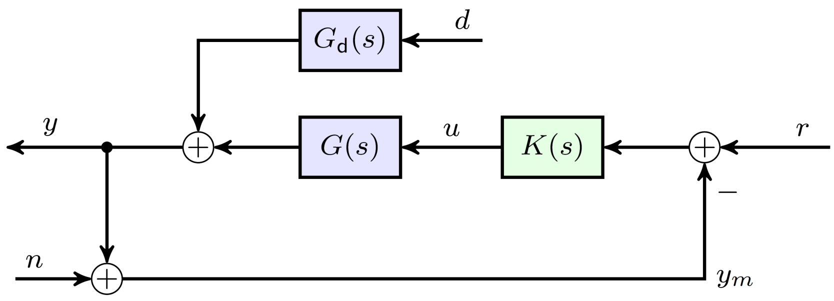
$$L(s) = \frac{\omega_c}{s}$$

This will give a phase margin of 90°.

$$K(s) = \frac{\omega_c}{s} G^{-1}(s)$$

Inverting the plant can often be done only approximately.

Design Example



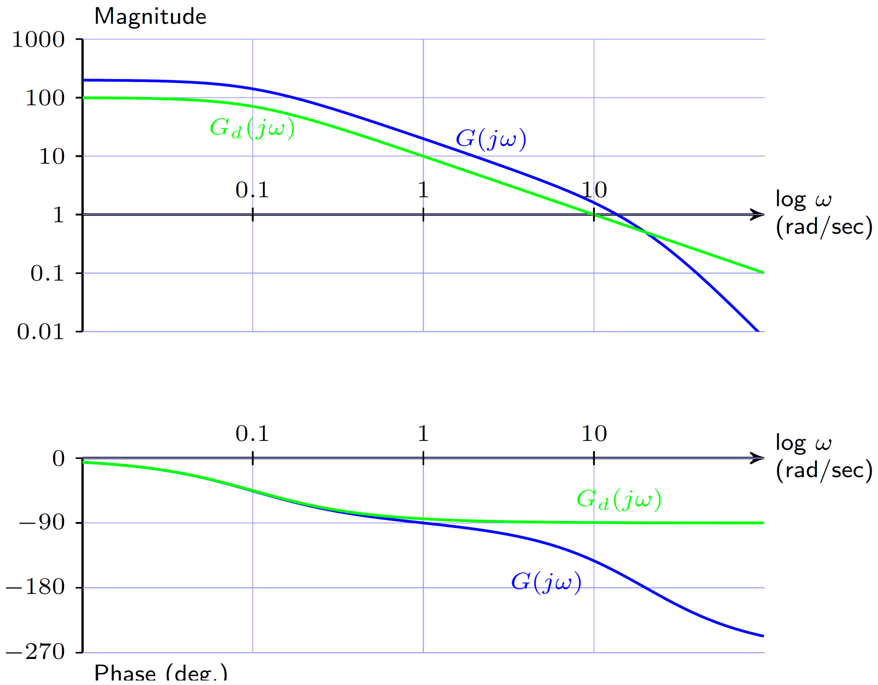
$$G(s) = \frac{200}{(10s+1)}\,\frac{1}{(0.05s+1)^2}, \qquad G_d(s) = \frac{100}{10s+1}$$

Objectives:

1. Rise time < 0.3 seconds.
2. Overshoot $< 5\%$
3. Disturbance response, $y_d(t)$, satisfies $|y(t)| \leq 1$.
4. Disturbance response, $y_d(t)$, satisfies $|y(t)| < 0.1$ within 3 seconds.
5. $|u(t)| \leq 1$ at all times.

$|G_d(j\omega)| > 1$ up to $\omega_d \approx 10$ rad/sec $\Longrightarrow$ $\omega_c \geq 10$ rad/sec.

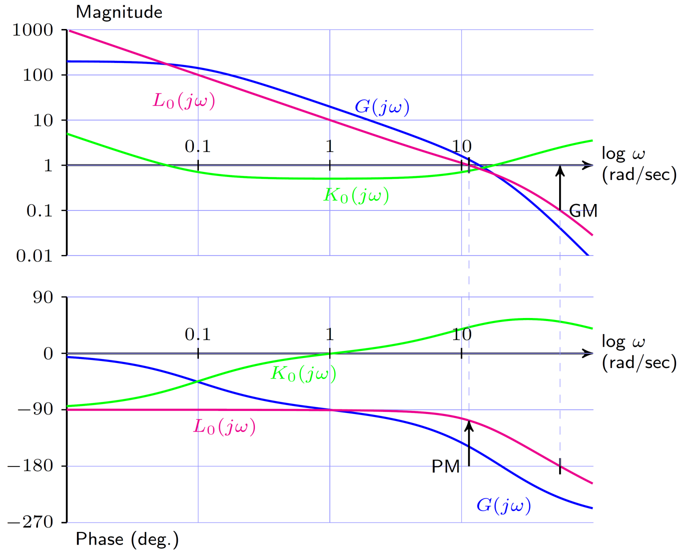
Closed-Loop Performance



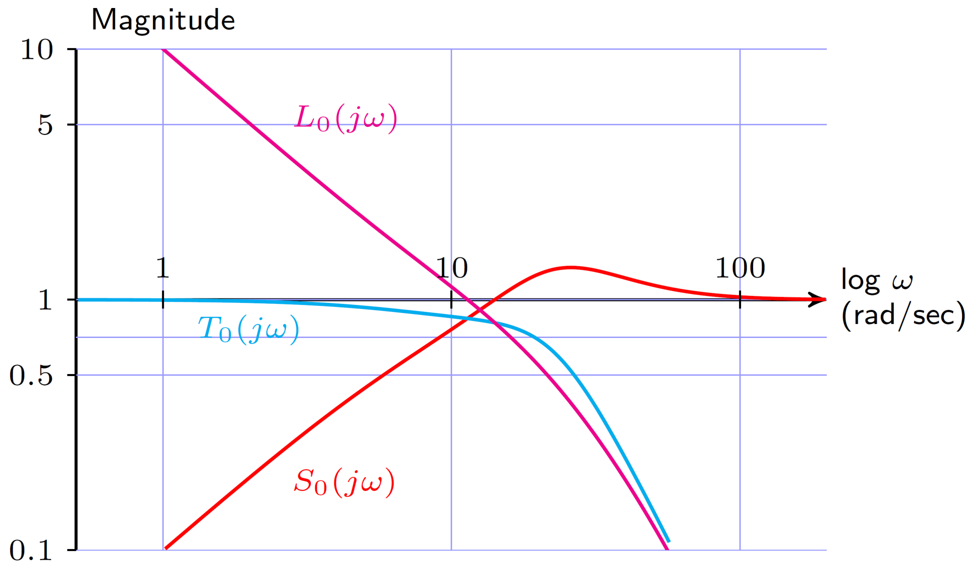
Inverse-based Control Design

The plant is stable and minimum-phase:

$$K_0(s) = \frac{\omega_c}{s} G^{-1}(s) = \frac{\omega_c}{s} \frac{(10s+1)}{200} (0.05s+1)^2 \approx \frac{\omega_c}{s} \frac{(10s+1)}{200} \frac{(0.1s+1)}{(0.01s+1)}$$



Inverse-based Control Design

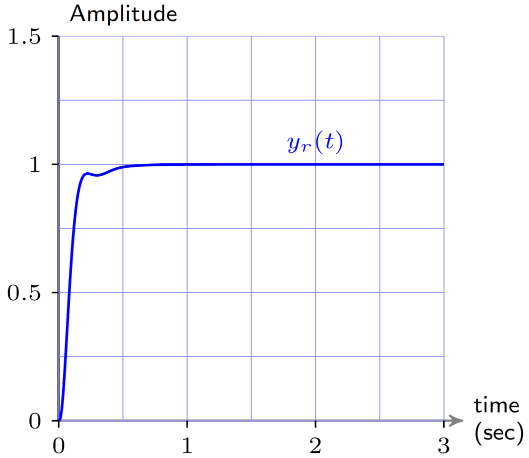


The closed-loop bandwidth is very close to $\omega_c \approx 10$ rad/sec.

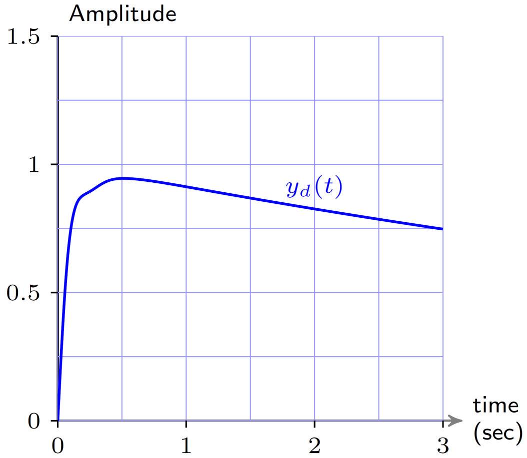
Inverse-based Control Design

$$K_0 = \frac{\omega_c}{s}\frac{(10s+1)}{200}\frac{(0.1s+1)}{(0.01s+1)}$$

Reference tracking



Disturbance response



Loop Shaping for Disturbance Rejection

Disturbance response: $y_d = SG_\mathsf{d}\, d + \ldots$

To achieve $|y_d(t)| \leq 1$ for $|d(t)| \leq 1$,

we want $|SG_\mathsf{d}(j\omega)| < 1$ for all ω.

So, we want:

$|1 + L(j\omega)| > |G_\mathsf{d}(j\omega)|$ for all ω.

or, approximately, $|L(j\omega)| > |G_\mathsf{d}(\omega)|$ for all ω.

Initial guess:

$|L_{\mathsf{min}}| \approx |G_\mathsf{d}|$ or $|K_{\mathsf{min}}| \approx |G^{-1}G_\mathsf{d}|$

Loop Shaping for Disturbance Rejection

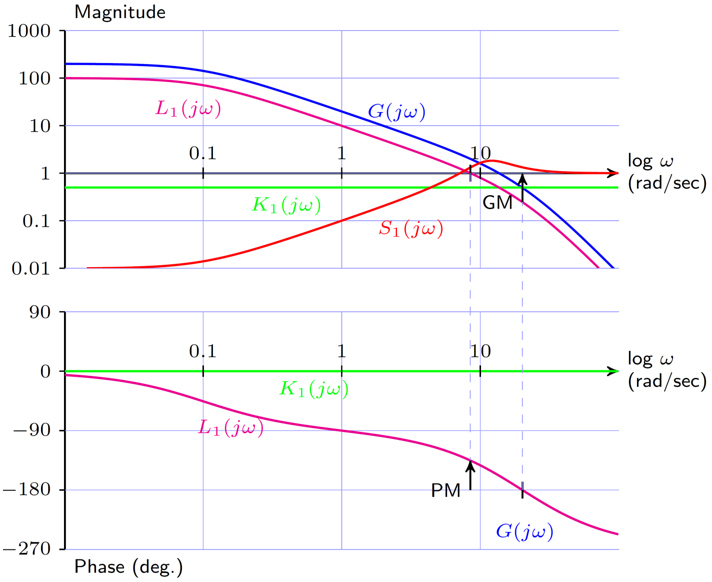
Step 1

Initial guess:

$$|K_{\mathsf{min}}| \approx |G^{-1}G_{\mathsf{d}}|$$

Choose:

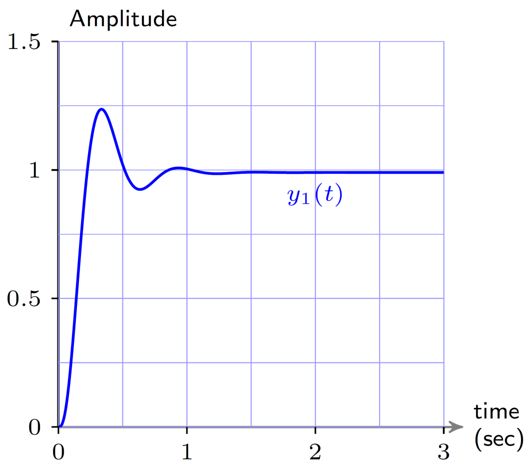
$$K_1(s) \approx G^{-1}(s)G_{\mathsf{d}}(s)$$
$$\approx 0.5(0.05s+1)^2$$
$$= 0.5$$



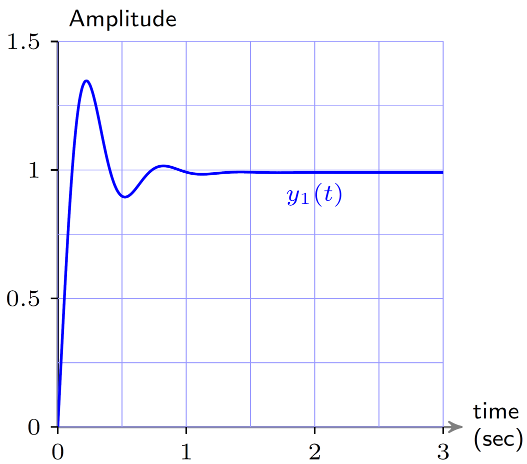
Loop Shaping for Disturbance Rejection

$$K_1 = 0.5 \quad \Longrightarrow \quad L_1(s) \approx G_{\mathsf{d}}(s)$$

Reference tracking



Disturbance response



Loop Shaping for Disturbance Rejection

Step 2 Increase the gain at low frequency.

To get integral action multiply the controller by:

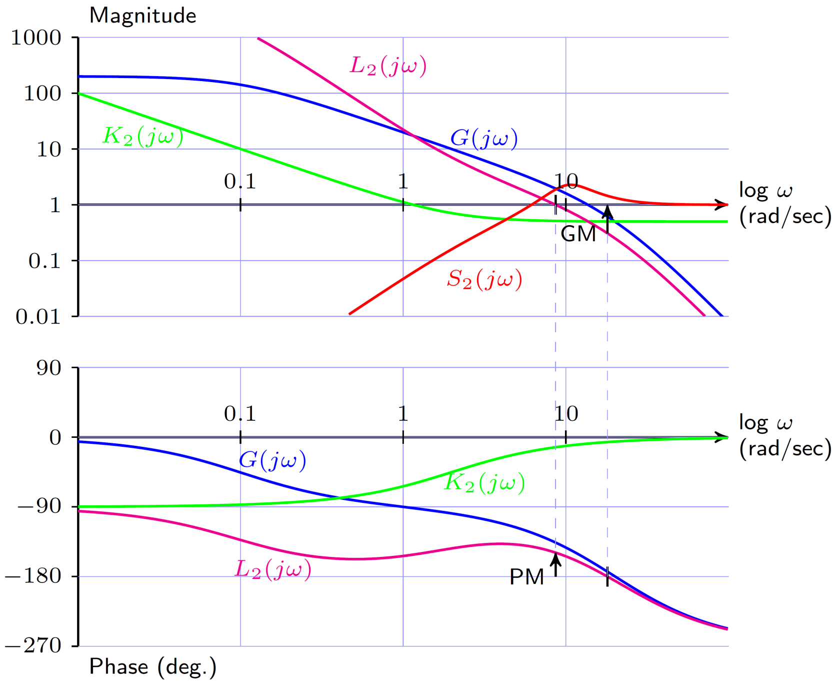
$$K_2(s) = \frac{s + \omega_I}{s} K_1(s).$$

If $\omega_I = 0.2\omega_c$ we get 11° more phase at ω_c than with K_1 alone.

So,

$$K_2(s) = 0.5\frac{(s + 2)}{s}$$

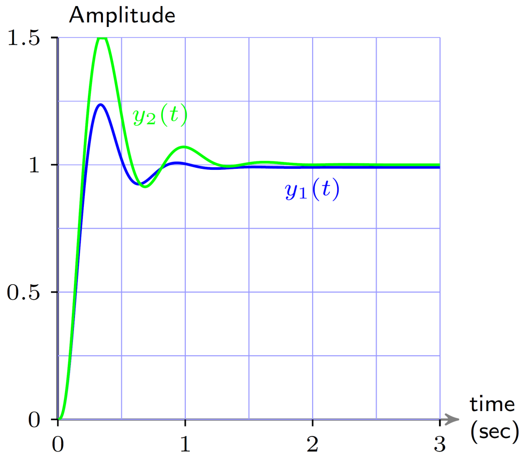
Loop Shaping for Disturbance Rejection



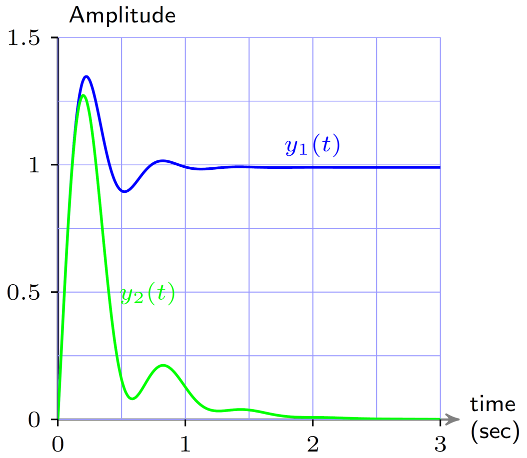
Loop Shaping for Disturbance Rejection

$$K_1 = 0.5 \quad \text{and} \quad K_2 = 0.5\frac{(s+2)}{s}$$

Reference tracking



Disturbance response



Loop Shaping for Disturbance Rejection

Step 2

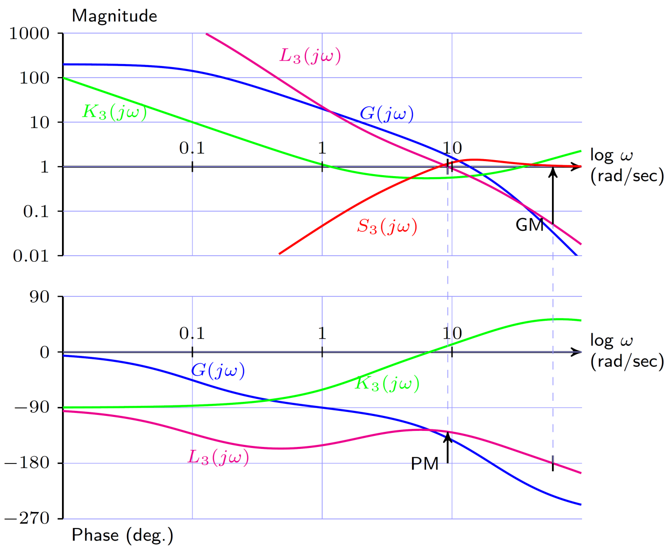
High frequency correction:

Augment with a lead-lag for "derivative action".

This will also improve the phase margin.

$$K_3(s) = K_2(s)\frac{(0.05s+1)}{(0.005s+1)}$$

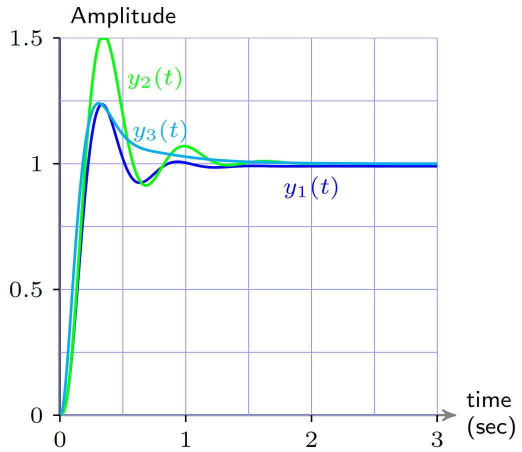
$$= 0.5\,\frac{(s+2)}{s}\,\frac{(0.05s+1)}{(0.005s+1)}$$



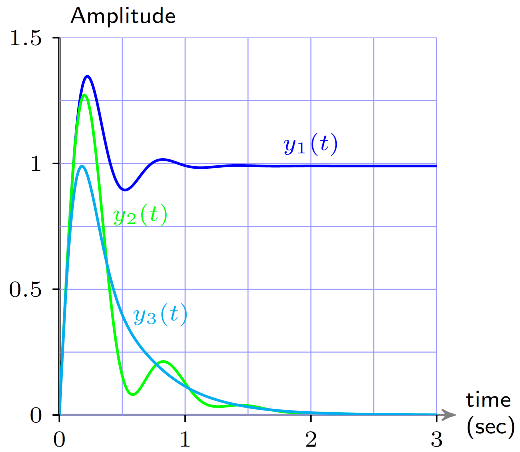
Loop Shaping for Disturbance Rejection

$$K_1 = 0.5, \qquad K_2 = 0.5\frac{(s+2)}{s}, \qquad K_3 = 0.5\frac{(s+2)}{s}\frac{(0.05s+1)}{(0.005s+1)}$$

Reference tracking

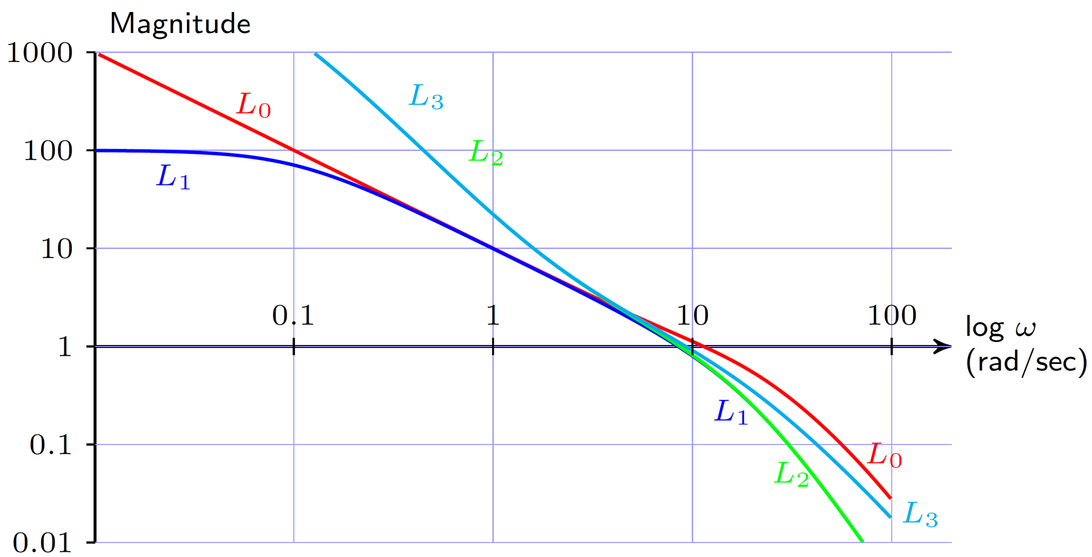


Disturbance response



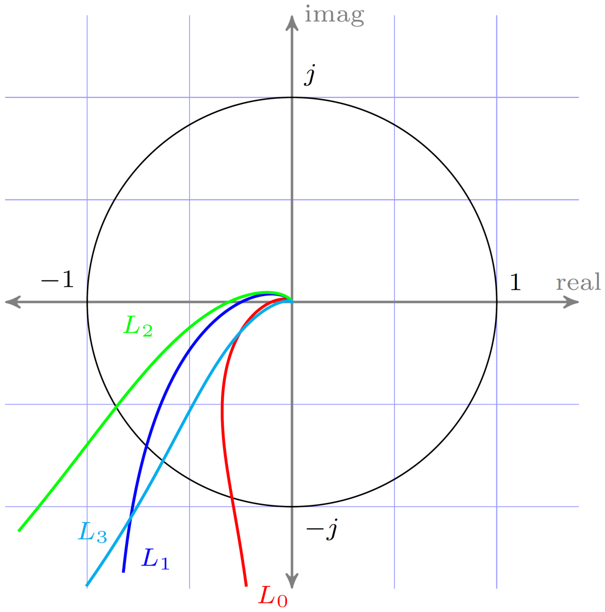
Loop Shaping for Disturbance Rejection

Loopshape comparisons



Loop Shaping for Disturbance Rejection

Loopshape comparisons



Project: Report 3

Consider your dynamic system. Using loop shaping concept design a controller K(s) to satisfy:

$$K_v \geq 10 \text{ and } \text{PM} \geq 40°$$

Deadline: The day before next Meeting

Please only use this email address: bevranih18@gmail.com

Thank You!