Robust Control Systems

Multivariable Control: An Introduction

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Reference

1. S. Skogestadand I. Postlethwaite, **Multivariable Feedback Control; Analysis and Design**, Second Edition, Wiley, 2005.

2. M. Fujita, **Lecture Notes on Feedback Control Systems**, Tokyo Institute of Technology, 2019.

3. R. Smith, **Lecture Notes on Control Systems**, ETH Zurich, 2020.

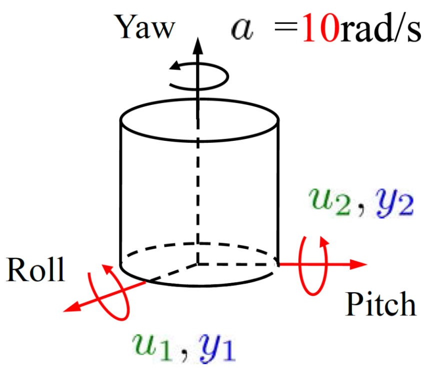
4. H. S. Tsien, **Engineering Cybernetics**, McGraw-Hill, 1954.

Robust Control of Multivariable Systems

Example: Spinning Satellite

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \begin{bmatrix} \frac{s-100}{s^2+100} & \frac{10s+10}{s^2+100} \\ \frac{-10s-10}{s^2+100} & \frac{s-100}{s^2+100} \end{bmatrix}$$

$$= \left[\begin{array}{cc|cc} 0 & 10 & 1 & 0 \\ -10 & 0 & 0 & 1 \\ \hline 1 & 10 & 0 & 0 \\ -10 & 1 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \qquad \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$



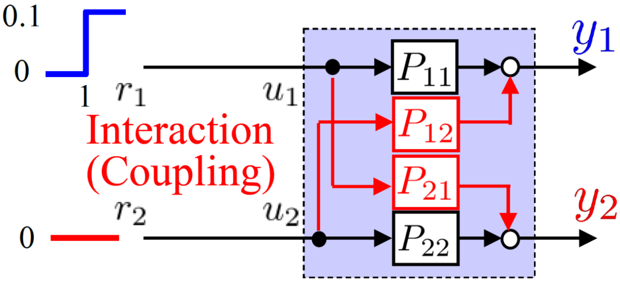
Inputs: u_1 u_2 Torque

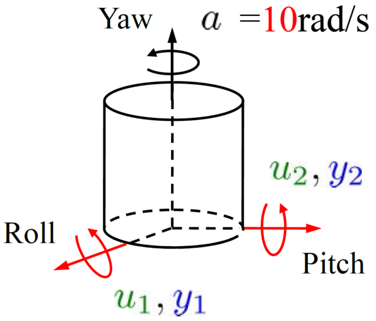
Outputs: y_1 y_2 Angular velocity

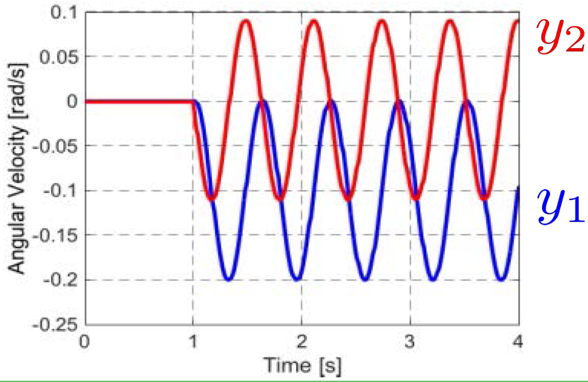
(Ref 4)

Spinning Satellite

$$P(s)=\begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix}$$







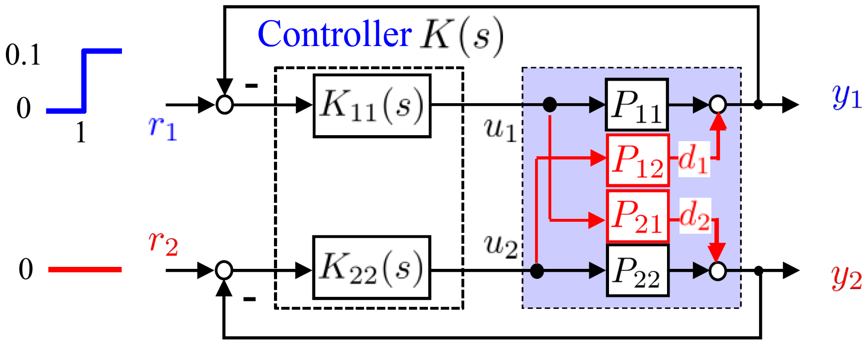
(Ref 4)

Robust Control of Multivariable Systems

1. Diagonal PID Controller (Decentralized Control)

$$P(s)=\begin{bmatrix} \frac{s-100}{s^2+100} & \frac{10s+10}{s^2+100} \\ \frac{-10s-10}{s^2+100} & \frac{s-100}{s^2+100} \end{bmatrix}$$

$$K(s)=\begin{bmatrix} K_{11}(s) & 0 \\ 0 & K_{22}(s) \end{bmatrix}$$



pidtune

(Ref 1, pp. 91-93)

Continue

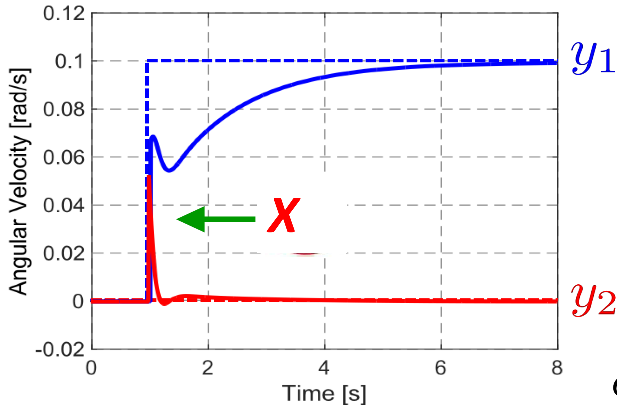
MATLAB Command

```
P11 = tf([1 -100],[1 0 100]) ;
K = pidtune( P11, 'PID') ;
```

$$P(s) = \begin{bmatrix} \frac{s-100}{s^2+100} & \frac{10s+10}{s^2+100} \\ \frac{-10s-10}{s^2+100} & \frac{s-100}{s^2+100} \end{bmatrix}$$

$$K(s) = \begin{bmatrix} K_{11}(s) & 0 \\ 0 & K_{22}(s) \end{bmatrix}$$

$$K_{11}(s) = K_{22}(s) = -0.864 - \frac{1.34}{s} - 0.135s$$



(Ref 1, pp. 91-93)

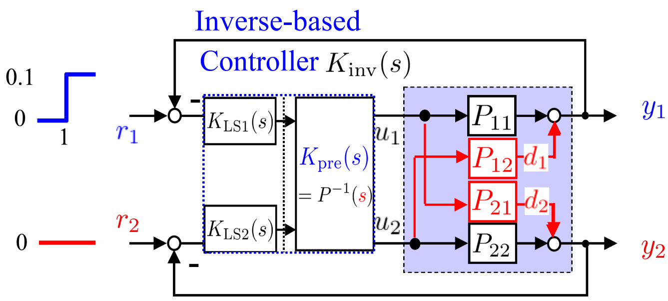
Robust Control of Multivariable Systems

2. Dynamic Decoupling $P^{-1}(s)$ (Inverse-based Controller $K_{\mathrm{inv}}(s)$)

$$P^{-1}(s) = 0.01 \begin{bmatrix} s-100 & -10s-10 \\ 10s+10 & s-100 \end{bmatrix}$$

$$P'(s) = P(s)P^{-1}(s) = I$$

$$K_{\mathrm{inv}}(s) = P^{-1}(s) \begin{bmatrix} K_{\mathrm{LS1}}(s) & 0 \\ 0 & K_{\mathrm{LS2}}(s) \end{bmatrix}$$

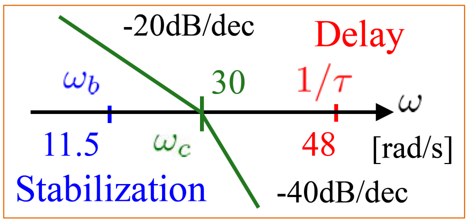


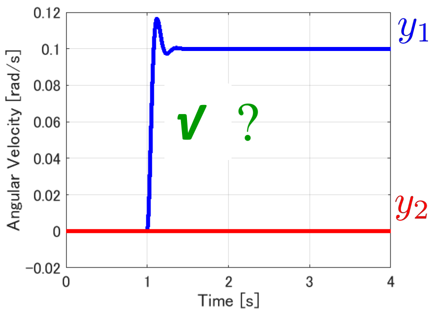
Loop Shaping Design

$$K_{\mathrm{LS1}}(s) = K_{\mathrm{LS2}}(s) = \frac{900}{s(s+30)}$$

Target Loop (Desired Loop)

$$L_{\mathrm{target}}(s) = \frac{900}{s(s+30)} I_2$$





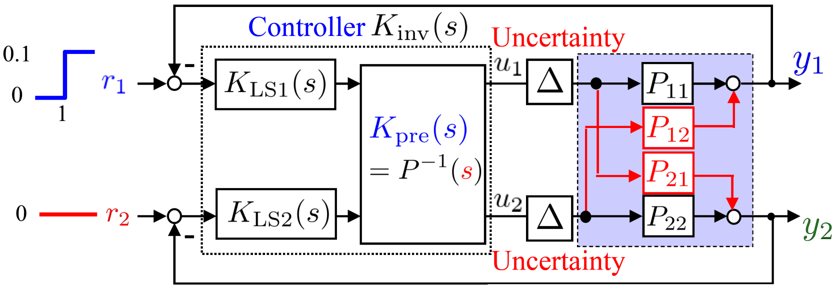
Robust Control of Multivariable Systems

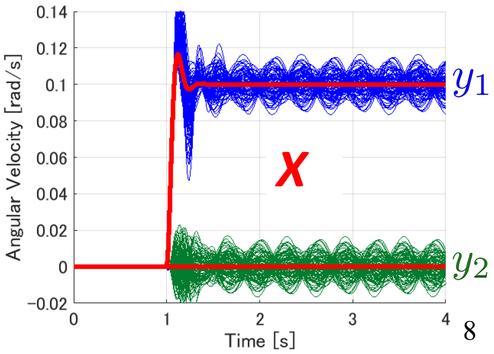
Impact of Uncertainty

$$P^{-1}(s) = 0.01 \begin{bmatrix} s-100 & -10s-10 \\ 10s+10 & s-100 \end{bmatrix}$$

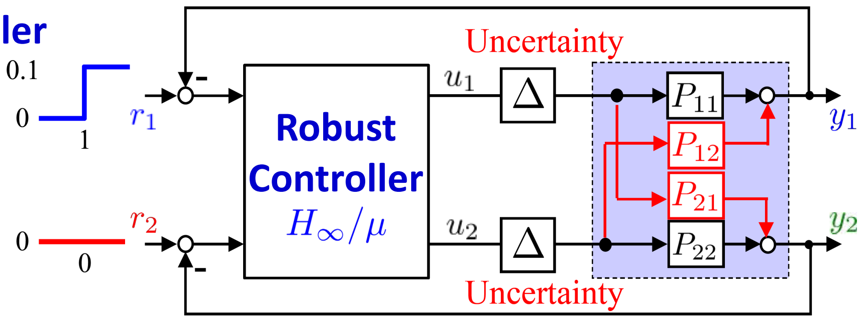
$$K_{\mathrm{LS1}}(s) = K_{\mathrm{LS2}}(s) = \frac{900}{s(s+30)}$$

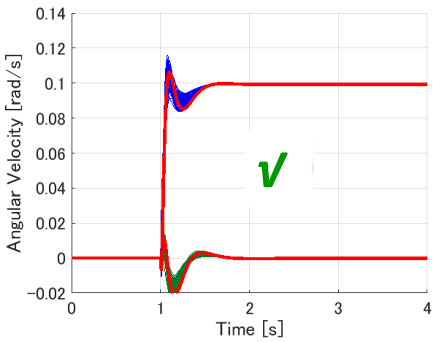
$$K_{\mathrm{inv}}(s) = P^{-1}(s) \begin{bmatrix} K_{\mathrm{LS1}}(s) & 0 \\ 0 & K_{\mathrm{LS2}}(s) \end{bmatrix}$$

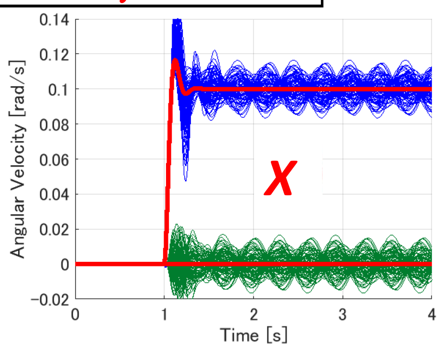




2. Robust Controller





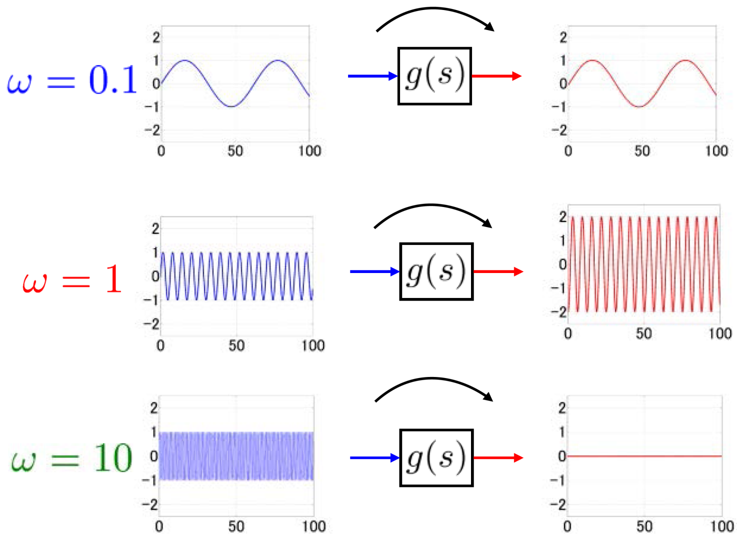


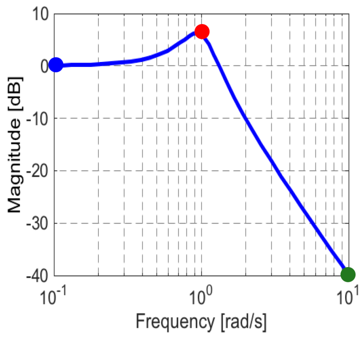
Frequency Response for SISO Systems

Example:

$$g(s) = \frac{1}{s^2 + 0.5s + 1} \quad \omega_n = 1 \quad \zeta = 0.25$$

$u(t) = \sin(\omega t)$ $\quad$ $y(t) = |g(j\omega)| \sin(\omega t + \angle g(j\omega))$





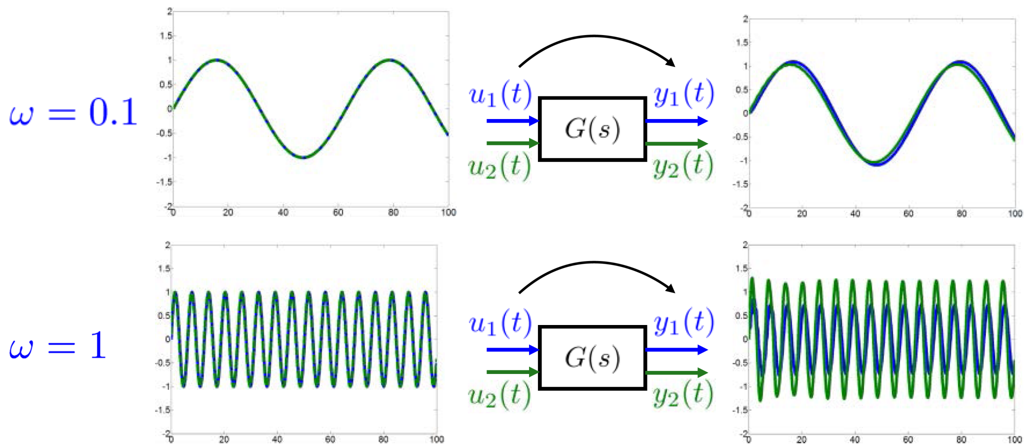
Bode Plot
(Gain)

Frequency Response for MIMO Systems

Example:

$$G(s) = \begin{bmatrix} \frac{10(s+1)}{s^2+0.2s+100} & \frac{1}{s+1} \\ \frac{s+2}{s^2+0.1s+10} & \frac{5(s+1)}{(s+2)(s+3)} \end{bmatrix}$$

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \qquad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$



Singular Value

Definition: The singular values of an $m \times n$ matrix A are the square roots of the non-zero eigenvalues of the symmetric $n \times n$ matrix A^TA or AA^T listed with their multiplicities in decreasing order $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$.

Example: Find the singular values for matrix A.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Solution: $$A^T A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\det(A^TA - \lambda I_2) = |A^TA - \lambda I_2| = (2-\lambda)^2 - 1 = 0. \Rightarrow \begin{matrix} \lambda_1 = 3 \\ \lambda_2 = 1 \end{matrix} \Rightarrow \boxed{\sigma_1 = \sqrt{3}} \quad \boxed{\sigma_2 = 1}$$

Singular Value Decomposition (SVD)

Theorem: Let **A** be an m × n matrix with $m \geq n$. Then there exist orthogonal matrices **U** (m×m) and V (n×n) and a diagonal matrix $\Sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_n)$ (m × n) with order $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$, such that **A** holds

$$A = U\Sigma V^T$$

The column vectors of **U** = [u1, . . . ,um] are called the **left singular vectors** and similarly **V** = [v1, . . . , vn] are the **right singular vectors** of matrix **A**. The columns of U and V are orthonormal. The matrix Σ is diagonal with positive real entries of σ_i , and can be represented as:

$$\Sigma = \begin{bmatrix} \tilde{\Sigma}_{l \times l} & 0_{l \times (n-l)} \\ 0_{(m-l) \times l} & 0_{(m-l) \times (n-l)} \end{bmatrix}$$

Where $l = \min(m, n)$ and

$$\tilde{\Sigma} = diag(\sigma_1, \sigma_2, \ldots, \sigma_l)$$

Singular Value Decomposition (SVD)

Example: $G = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$ ⇨ $G = \begin{bmatrix} -0.87 & -0.48 \\ -0.48 & 0.87 \end{bmatrix} \begin{bmatrix} \underline{7.34} & 0 \\ 0 & \underline{0.27} \end{bmatrix} \begin{bmatrix} -0.79 & -0.60 \\ -0.60 & -0.79 \end{bmatrix}$

$$G = U \Sigma V^H$$

$y = Gu$

U, V : Unitary Matrices

Singular Values $\sigma_1, \cdots, \sigma_p$

$(\sigma_1 > \sigma_2 > \cdots > \sigma_p)$

$\sigma_i = \sqrt{\lambda_i(G^H G)}$ λ_i : i -th eigenvalue

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \sigma_p & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Maximum Singular Value

$$\bar{\sigma}(G) = \sigma_1 = \max_{u \neq 0} \|y\|_2 / \|u\|_2$$

Minimum Singular Value

$$\underline{\sigma}(G) = \sigma_p = \min_{u \neq 0} \|y\|_2 / \|u\|_2$$

(Ref 1, A.3)

SVD

Matlab function: `> [U, S, V ] = svd(A)`

If σ_r is the smallest singular value greater than zero then the matrix A has rank r, and $s_r > 0$. In this case U and V can be partitioned as U=[U1,U2] and V=[V1,V2], where $U_1 = [u_1, u_2, \ldots, u_r)$ and $V_1 = [v_1, v_2, \ldots, v_r)$ have r columns. Then A can be represented as ***reduced*** form of SVD as follows

$$A = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}^T = U_1 \Sigma_r V_1^T = \sum_{i=1}^{r} u_i v_i^T \sigma_i$$

`> [U, S, V ] = svd(A,0)`

Example: Find full and reduced SVD for matrix A.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

SVD

Solution:

```
>> A=[0 1;1 1;1 0]
A =
     0     1
     1     1
     1     0

>> [U,S,V] = svd(A)
U =
   -0.4082    0.7071    0.5774
   -0.8165    0.0000   -0.5774
   -0.4082   -0.7071    0.5774

 S =
    1.7321         0
         0    1.0000
         0         0
 V =
   -0.7071   -0.7071
   -0.7071    0.7071
```

```
>> [U,S,V] = svd(A,0)

U =
   -0.4082    0.7071
   -0.8165    0.0000
   -0.4082   -0.7071
S =
    1.7321         0
         0    1.0000
V =
   -0.7071   -0.7071
   -0.7071    0.7071
```


Continue

Example:

$$G(s) = \begin{bmatrix} \frac{10(s+1)}{s^2+0.2s+100} & \frac{1}{s+1} \\ \frac{s+2}{s^2+0.1s+10} & \frac{5(s+1)}{(s+2)(s+3)} \end{bmatrix}$$

$u(s) \rightarrow G(s) \rightarrow y(s)$

$$y(\omega) = G(j\omega)u(\omega)$$

σ-plot is the extension of Bode gain plot to MIMO Systems

SISO: Absolute value

$$|g(j\omega)|$$

MIMO: Singular value plot

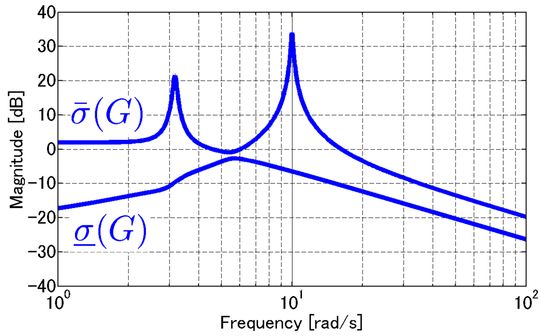
$$\sigma(G(j\omega))$$

MATLAB Command

```
num = { [10 10], 1;
        [1 2],   [5 5] };
den = { [1 0.2 100], [1 1];
        [1 0.1 10],   [1 5 6] };
G = tf( num, den );

figure
sigma(G)
```

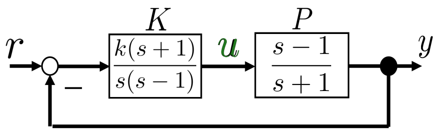
σ-plot of G



(Ref 1, p. 79)

Internal Stability

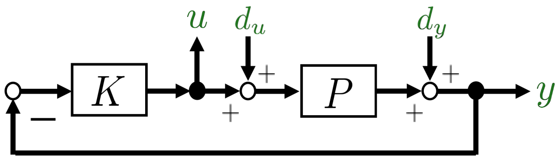
Example (SISO):



$r \rightarrow y \quad G_{yr} = \frac{k}{s+k} \quad (k>0)$ Stable | $r \rightarrow u \quad G_{ur} = \frac{k(s+1)}{(s-1)(s+k)}$ Unstable

$L = PK = \frac{s-1}{s+1}\frac{k(s+1)}{s(s-1)} = \frac{k}{s}$ Unstable pole/zero cancellation

In order to avoid pole/zero cancellation, consider input injection and output measurement for each dynamic block (Gang of Four).



Sensitivity: $\underset{(d_y \rightarrow y)}{S} = \frac{1}{1+PK}$

Complementary Sensitivity: $\underset{(d_u \rightarrow u)}{T} = \frac{PK}{1+PK}$

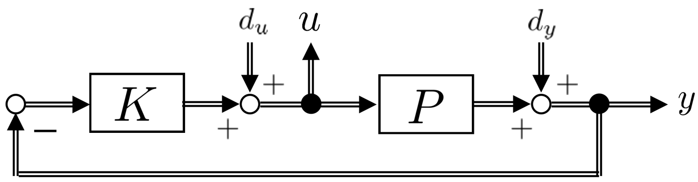
Load Sensitivity: $\underset{(d_u \rightarrow y)}{PS} = \frac{P}{1+PK}$

Noise Sensitivity: $\underset{(d_y \rightarrow u)}{KS} = \frac{K}{1+PK}$

(Ref 1, p. 144)

Internal Stability

Nominal Stability:



u, d_u, y, d_y : Vectors

P, K : Transfer function matrices

Well-posedness: $1 + P(\infty)K(\infty) \neq 0$

(Gang of Four: well-defined and proper)

$$u = (I+KP)^{-1}d_u - K(I+PK)^{-1}d_y$$
$$y = P(I+KP)^{-1}d_u + (I+PK)^{-1}d_y$$

Theorem: Assume P, K contain no unstable hidden modes. Then, the feedback system in the figure is ***internally stable*** if and only if all four closed-loop transfer matrices are stable.

Theorem: Assume $\left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]$ shows the state space representation of above system. The system is **internally stable** if and only if $\bar{A}$ is stable.

(Ref 1, p. 145)

Relative Gain Array (RGA)

Example:

$$y_1 = u_1 + u_2$$
$$y_2 = 0.4u_1 - 0.1u_2$$

$$\Rightarrow \quad G = \begin{bmatrix} 1 & 1 \\ 0.4 & -0.1 \end{bmatrix}$$

$$\text{RGA}(G) = \Lambda(G) := G \times (G^{-1})^T = \begin{bmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{bmatrix}$$

element wise multiplication

Pairing rule 1 Prefer paring on RGA elements close to 1

Use u_2 to control y_1 and use u_1 to control y_2

Pairing rule 2 Avoid pairing on negative RGA elements

Pairing rule 2 is satisfied for this choice

$$y_1 = u_1 + u_2$$
$$y_2 = -0.1u_1 + 0.4u_2$$

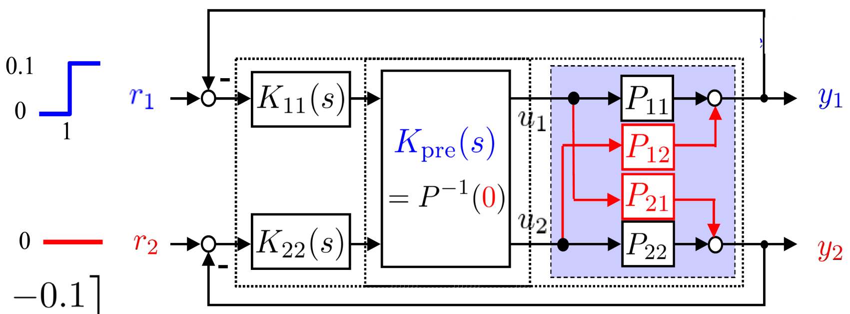
$$\Lambda = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

Rule 1 √
Rule 2 √

(Ref 1, Sec. 3.4, p. 85)

Control of Multivariable Systems

Steady-State Decoupling $P^{-1}(0)$

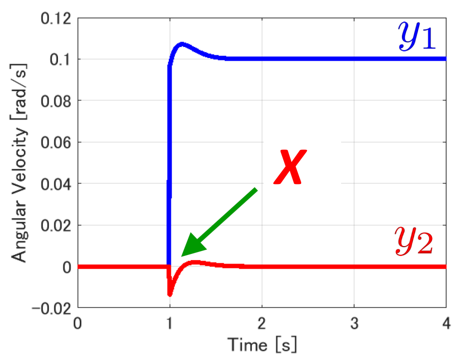


$$P'(s) = P(s)P^{-1}(0) = P(s)\begin{bmatrix} -1.0 & -0.1 \\ 0.1 & -1.0 \end{bmatrix}$$

$$P'(0) = I$$

$$K_{11}(s) = K_{22}(s) = 11.7 + \frac{49.1}{s} + 0.7s$$

$$K(s) = P^{-1}(0)\begin{bmatrix} K_{11}(s) & 0 \\ 0 & K_{22}(s) \end{bmatrix}$$



(Ref 1, pp. 91-93)

Poles

Theorem: The pole polynomial $\phi(s)$ corresponding to a minimal realization of a system with transfer function $G(s)$ is the least common denominator of all non-identically zero minors of all orders of $G(s)$.

Example: $$G(s) = \begin{bmatrix} \frac{1}{s+1} & 0 & \frac{s-1}{(s+1)(s+2)} \\ \frac{-1}{s-1} & \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix}$$

The minors of order 1: $M^2_{23} = \det \begin{bmatrix} \frac{1}{s+1} & 0 & \frac{s-1}{(s+1)(s+2)} \\ \frac{-1}{s-1} & \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix} = \frac{1}{s+1}$

$M^2_{12} = \frac{s-1}{(s+1)(s+2)}$ $M^1_{23} = \frac{-1}{s-1}$

$M^1_{13} = \frac{1}{s+2}$ $M^1_{12} = \frac{1}{s+2}$

The minors of order 2: $M_2 = \det \begin{bmatrix} \frac{1}{s+1} & 0 & \frac{s-1}{(s+1)(s+2)} \\ \frac{-1}{s-1} & \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix} = \frac{2}{(s+1)(s+2)}$

$M_1 = \frac{-(s-1)}{(s+1)(s+2)^2}$

$M_3 = \frac{1}{(s+1)(s+2)}$

The least common denominator of all the minors:

$\phi(s) = (s+1)(s+2)^2(s-1)$ ➡ Poles $p = 1, -1, -2, -2$

(Ref 1, p. 135)

Zeros

Theorem: The zero polynomial $z(s)$, corresponding to a minimal realization of the system, is **the greatest common divisor** of all the numerators of all order-r minors of $G(s)$, where is the normal rank of $G(s)$, provided that these minors have been adjusted in such a way as to have the pole polynomial $\phi(s)$ as their denominator.

Example: $$G(s) = \begin{bmatrix} \frac{1}{s+1} & 0 & \frac{s-1}{(s+1)(s+2)} \\ \frac{-1}{s-1} & \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix}$$

Normal rank: 2

$$\phi(s) = (s+1)(s+2)^2(s-1)$$

The minors of order 2:

$M_1 = \frac{-(s-1)}{(s+1)(s+2)^2} = \frac{-(s-1)^2}{\phi(s)}$

$M_3 = \frac{1}{(s+1)(s+2)} = \frac{(s-1)(s+2)}{\phi(s)}$

$M_2 = \frac{2}{(s+1)(s+2)} = \frac{2(s-1)(s+2)}{\phi(s)}$

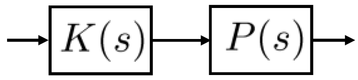
The greatest common divisor of numerator:

$z(s) = (s-1)$ ➡ Zeros $z = 1$

(Ref 1, p. 139)

Pole/Zero Cancellation

$$P(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{s+1} & \frac{1}{s+4} \end{bmatrix} \qquad K(s) = \begin{bmatrix} 1 & 0 \\ -\frac{s+4}{s+1} & \frac{1}{s+2} \end{bmatrix}$$



Poles $p = -1, -4$ Poles $p = -1, -2$

Poles of $P(s)$ and $K(s)$: $-1, -1, -2, -4$

$$L(s) = PK = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{(s+2)(s+4)} \end{bmatrix}$$ Poles $p = -1, -2, -4$

Poles of $P(s)$, $p = -1$ is cancelled

(Ref 1, Sec. 4.5)

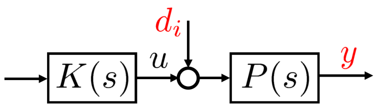
Sensitivity as a Feedback Performance Index

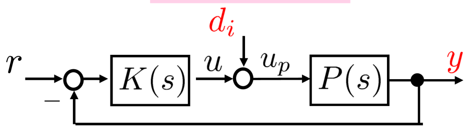
Disturbance Attenuation in SISO Systems

Open-loop **Closed-loop**

$d_i \to y$

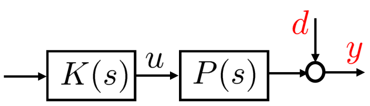
$$y(s) = P(s)d_i(s) \qquad y(s) = \frac{1}{1+P(s)K(s)}P(s)d_i(s)$$

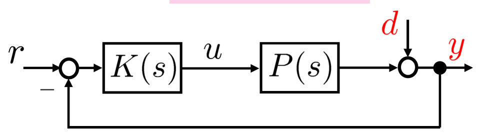




$d \to y$

$$y(s) = d(s) \qquad y(s) = \frac{1}{1+P(s)K(s)}d(s)$$

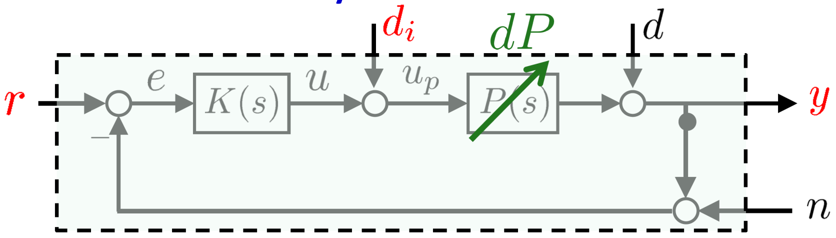




$$S(s) = \frac{1}{1+P(s)K(s)}$$ Small $|S(j\omega)|$ is a good feedback performance

Sensitivity as a Feedback Performance Index

Insensitivity to Plant Variations in SISO Systems



$$G_{yr} = \frac{PK}{1+PK} \quad (r \to y) \Rightarrow \frac{dG_{yr}}{G_{yr}} = S\frac{dP}{P}$$

$$\left(\frac{dG_{yr}}{dP} = \frac{K}{(1+PK)^2} = \frac{SPK}{P(1+PK)} = S\frac{G_{yr}}{P}\right)$$

$$G_{yd_i} = \frac{P}{1+PK} \quad (d_i \to y) \Rightarrow \frac{dG_{yd_i}}{G_{yd_i}} = S\frac{dP}{P}$$

$$\left(\frac{dG_{yd_i}}{dP} = \frac{1}{(1+PK)^2} = \frac{SP}{P(1+PK)} = S\frac{G_{yd_i}}{P}\right)$$

Small absolute value of S ($|S(j\omega)|$) is a good feedback performance

(Ref 1, p. 23)

H∞ Norm as a System Gain

In MIMO Systems, System Gain:

$$\|G(s)\|_\infty = \max_\omega \bar{\sigma}(G(j\omega))$$

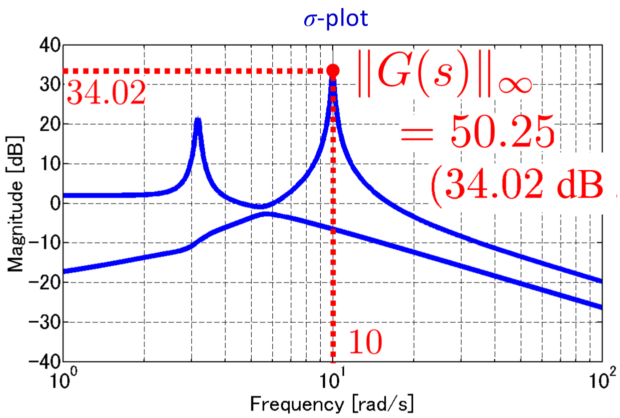
$G(s) \in \mathcal{S}$: Proper stable system

Example: $G(s) = \begin{bmatrix} \frac{10(s+1)}{s^2+0.2s+100} & \frac{1}{s+1} \\ \frac{s+2}{s^2+0.1s+10} & \frac{5(s+1)}{(s+2)(s+3)} \end{bmatrix}$

MATLAB Command

hinfG = normhinf(G)

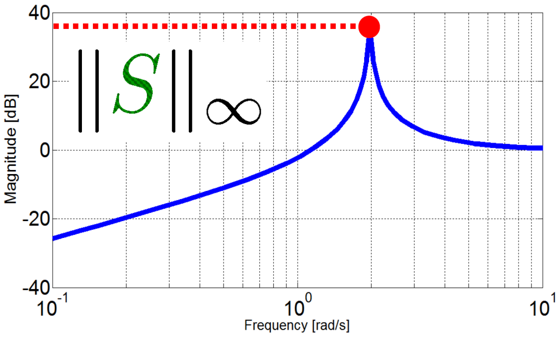
$$\|G(s)\|_\infty = \max_{\|\omega\|=1} \|z\|_2 = \max_{\omega \neq 0} \frac{\|z\|_2}{\|\omega\|_2}$$



(Ref 1, p. 158)

Difference Between the H∞ and H2 Norms

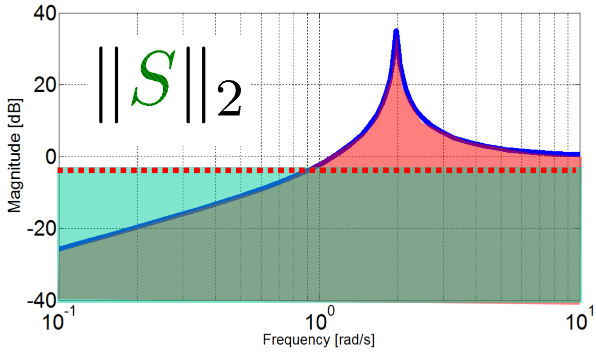
Minimizing H∞ norm: Push down "peak of maximum singular value"



Multiplicative property:

$$\|A(s)B(s)\|_\infty \leq \|A(s)\|_\infty \|B(s)\|_\infty$$

Minimizing H2 norm: Push down "whole thing (all singular values over all frequencies)"



Multiplicative property:

$$\|A(s)B(s)\|_2 \overset{?}{\leq} \|A(s)\|_2 \|B(s)\|_2$$

(Ref 1, pp. 75, 159)

Thank You!