Robust Control Systems

Uncertainty and Robust Stability

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Fall 2023

Contents

Reference

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3. R. Smith, **Lecture Notes on Control Systems**, ETH Zurich, 2020.

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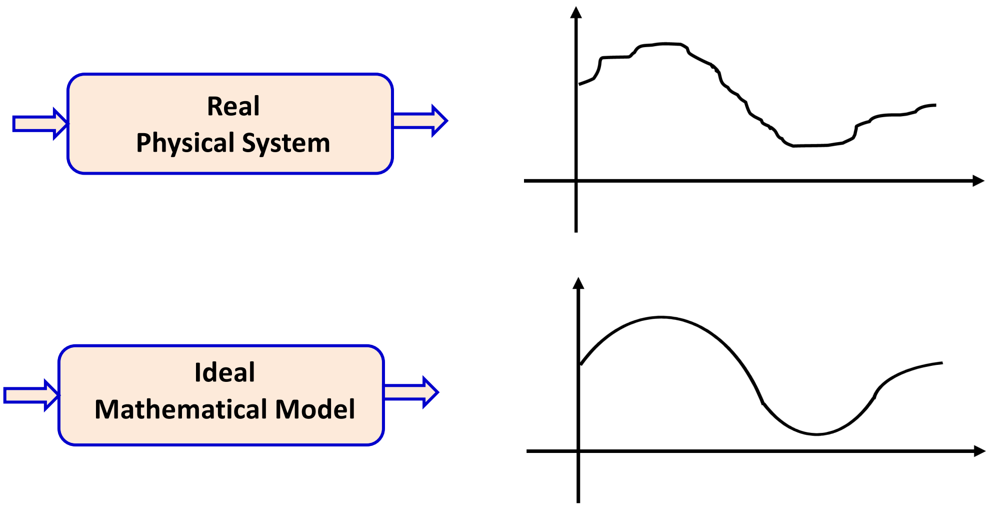
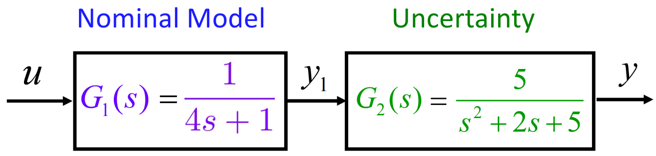
Introduction

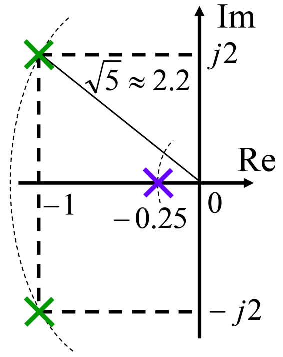


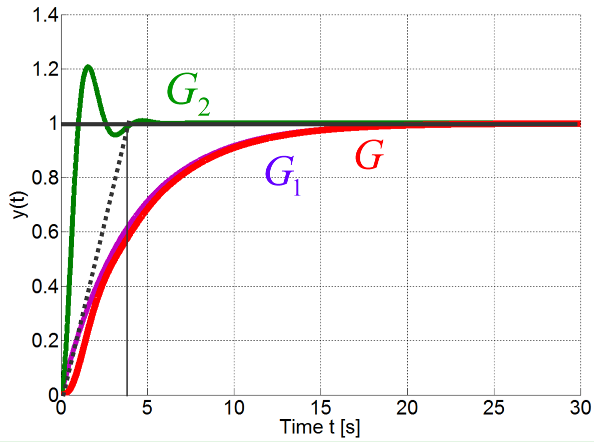
Example (Modeling Uncertainty)

$$G(s) = G_1(s) \cdot G_2(s) \approx G_1$$
$$= \frac{1}{4s+1} \cdot \frac{5}{s^2+2s+5}$$



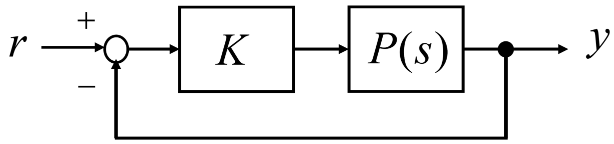
$$s = -0.25 \qquad s = -1 \pm j2$$



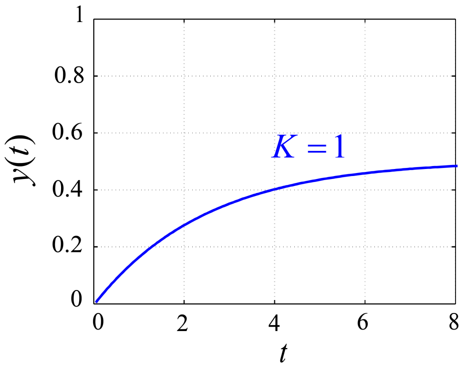


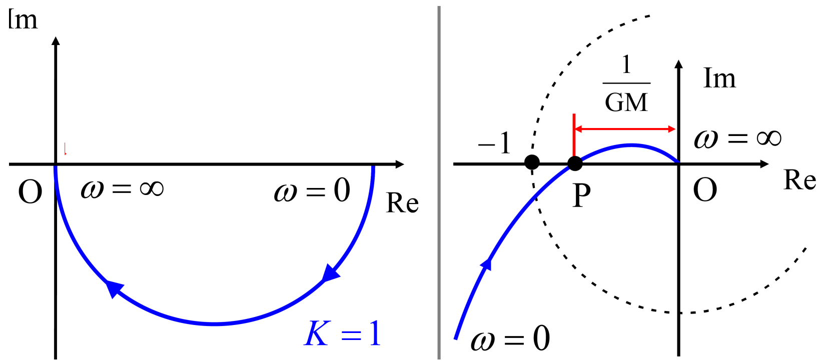
Example

$$P(s) = \frac{1}{5s+1} \qquad L(s) = \frac{K}{5s+1} = P(s)K$$



$$K = 1 \qquad L(s) = \frac{1}{5s+1}$$

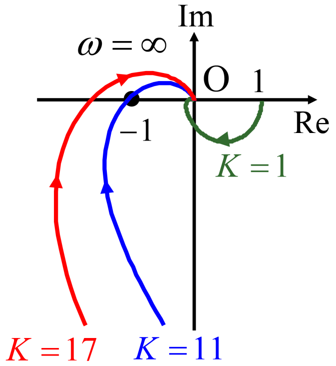


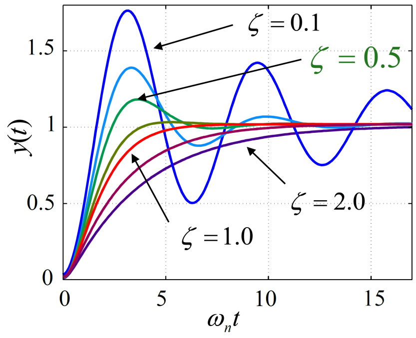


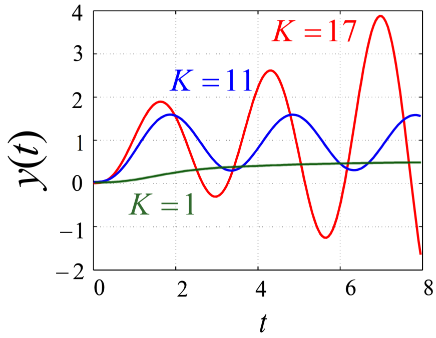
Example: With Oscillation Mode (Uncertainty)

$$P(s) = \frac{1}{5s+1} \Rightarrow \tilde{P}(s) = \frac{1}{5s+1} \cdot \frac{4}{s^2+2s+4}$$

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \Rightarrow \zeta = 0.5,\ \omega_n = 2$$

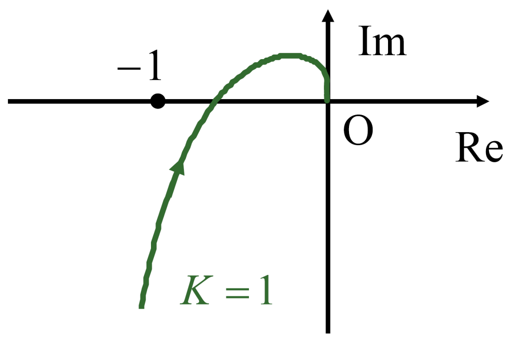




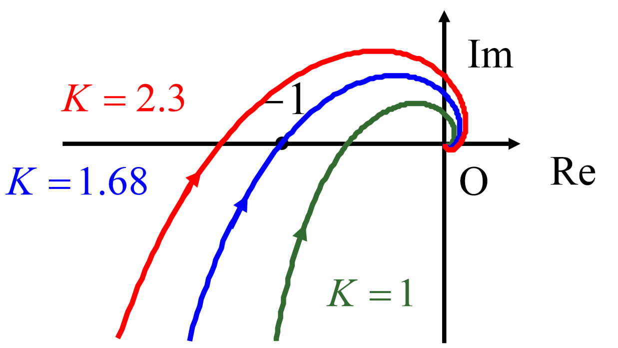


System with Uncertainty

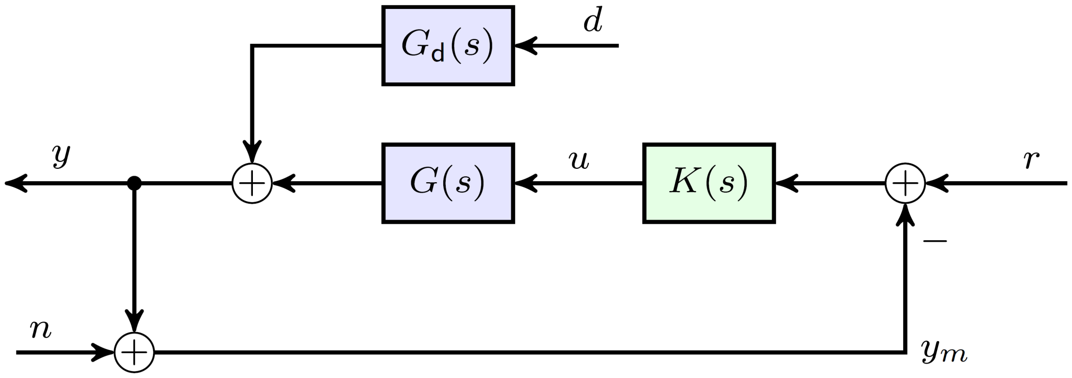
Nominal Model: $L(s) = \frac{K}{s(s+1)(s+2)}$



Uncertain Model: $\tilde{L}(s) = \frac{K e^{-s}}{s(s+1)(s+2)}$



Feedback Control



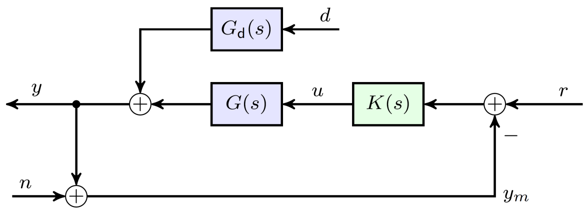
Objectives:

- Closed-loop stability
- Reference tracking
- Disturbance rejection
- Noise response

Difficulties:

- Model errors
- Fundamental limits on controllability of $G(s)$
- Actuation constraints

Transfer Functions



Loop transfer function

$$L(s) = G(s)K(s)$$

Sensitivity function

$$S(s) = \frac{1}{1 + G(s)K(s)} = \frac{1}{1 + L(s)}$$

Complementary sensitivity

$$T(s) = \frac{G(s)K(s)}{1 + G(s)K(s)} = \frac{L(s)}{1 + L(s)}$$

Output response

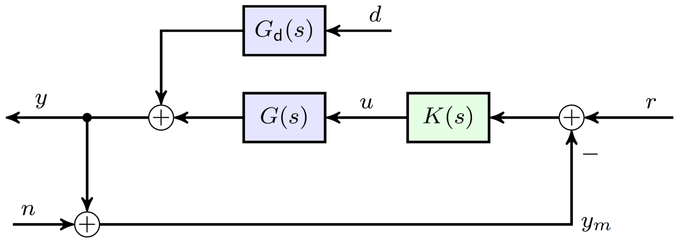
$$y = T(s)\,r \; + \; S(s)G_d(s)\,d \; - \; T(s)n$$

Error response

$$\begin{aligned} e &= r - y \\ &= S(s)\,r \; - \; S(s)G_d(s)\,d \; + \; T(s)n \end{aligned}$$

Conflicting Objectives

$$y = T(s)\,r \;+\; S(s)G_d(s)\,d \;-\; T(s)n$$

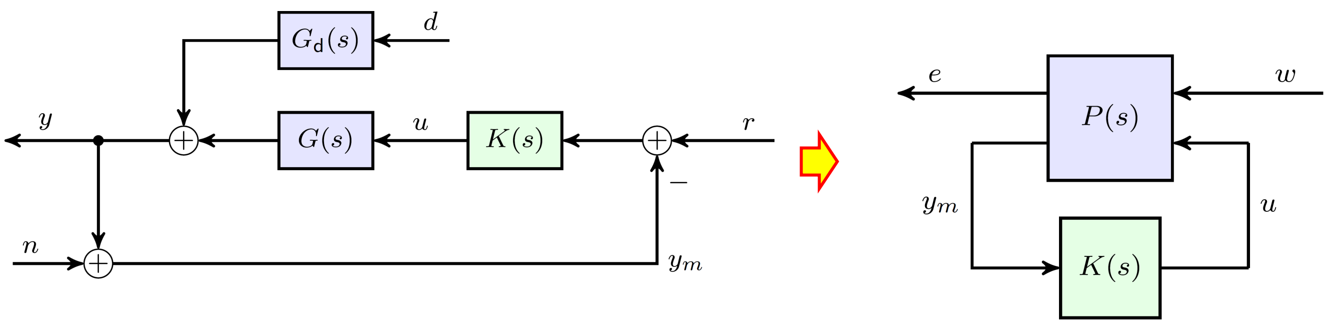


- **Performance Requirements**

Reference tracking	$T(s) \approx 1$
Noise rejection	$T(s) \ll 1$
Disturbance rejection	$S(s)G_d(s) \ll 1$
Low closed-loop plant sensitivity	$S(s) \ll 1$

- **Constraint** $S(s) + T(s) = 1$ for all s

Alternative Structures

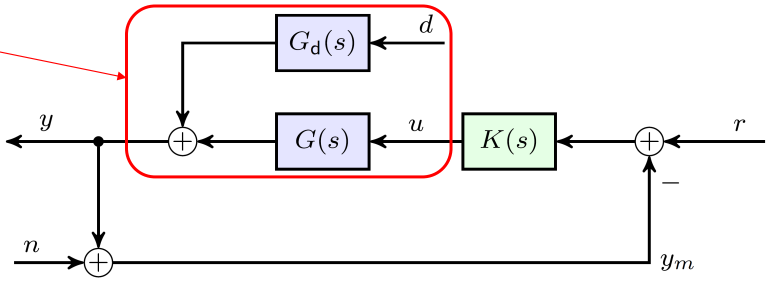


e = performance outputs w = exogenous inputs

y_m = controller inputs u = control actuation

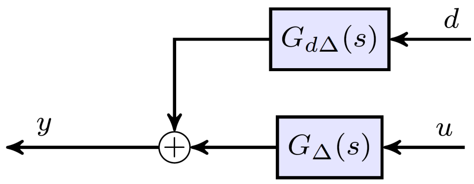
Perturbed Systems

Nominal case:



Sources of uncertainty:

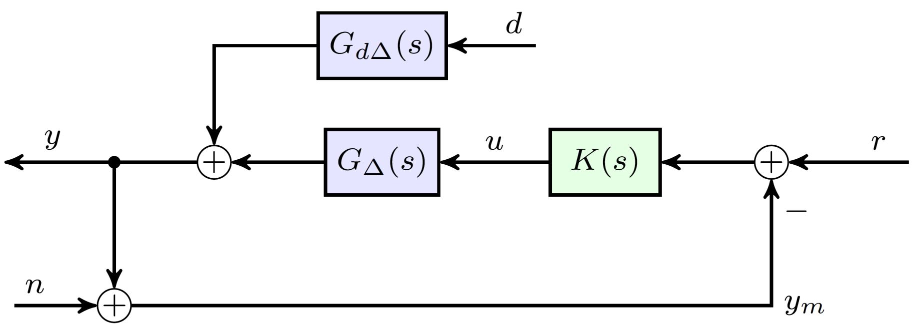
- Nonlinear dynamics.
- Operating point variation.
- Neglected dynamics in the model.
- Non-repeatable dynamics.



$$G_{\Delta}(s) = \{\, G(s) + \Delta(s) \mid \Delta(s) \in \text{ Set } \,\}$$
$$G_{d\Delta}(s) = \{\, G_d(s) + \Delta_d(s) \mid \Delta_d(s) \in \text{ Set } \,\}$$

Robustness

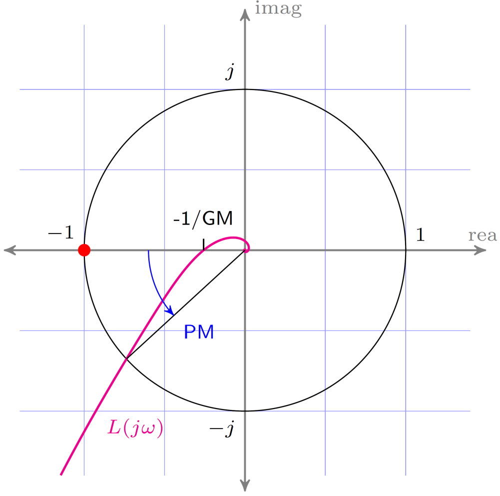
Closed-loop configuration:



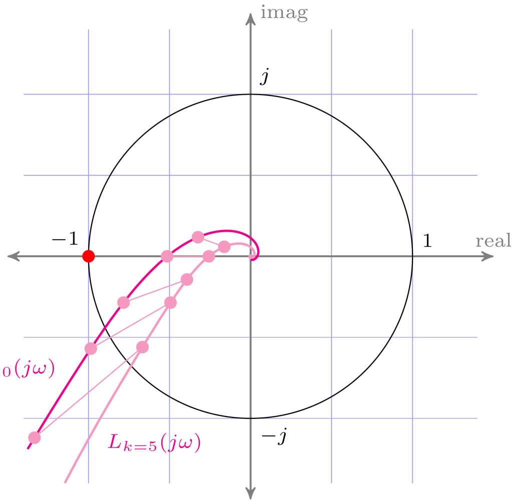
$$G_{\Delta}(s) = \{\, G(s) + \Delta(s) \mid \Delta(s) \in \text{ Set } \,\}$$
$$G_{d\Delta}(s) = \{\, G_d(s) + \Delta_d(s) \mid \Delta_d(s) \in \text{ Set } \,\}$$

What happens for the different Δ that may occur in practice?

Nyquist Plot



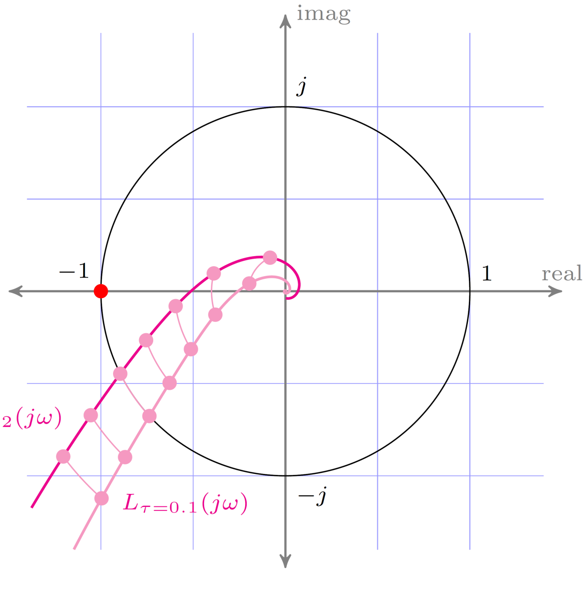
Margins (k=5)



Gain perturbation

$$G(s) = \frac{k\,\mathrm{e}^{-0.1s}}{(s+1)(0.1s+1)}, \quad k \in [5, 10] \qquad K(s) = \frac{0.5s+1}{s}$$

Nyquist Plot: Delay Perturbation



$$G(s) = \frac{5\,\mathrm{e}^{-\tau s}}{(s+1)(0.1s+1)}, \quad \tau \in [0.1, 0.2] \qquad K(s) = \frac{0.5s+1}{s}$$

Robustness Objectives and Approaches

Nominal Stability (NS)
Closed-loop system stable with no model uncertainty.

Nominal Performance (NP)
Closed-loop system satisfies the performance requirements with no model uncertainty.

Robust Stability (RS)
Closed-loop system is stable for all models in a prescribed set.

Robust Performance (RP)
Closed-loop system satisfies the performance requirements for all models in a prescribed set.

Some Approaches:

Loop shaping
Design $K(s)$ so that the loop, $L(s)$, has the required properties (classical approach).

Signal-based optimal control
Design $K(s)$ to satisfy certain closed-loop system or signal objectives. For example: LQG methods.

Numerical optimisation-based
Use multi-objective optimisation with closed-loop and robustness objectives.

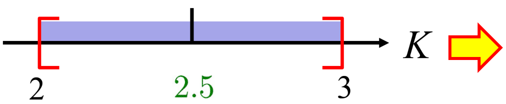
Uncertainty Modeling

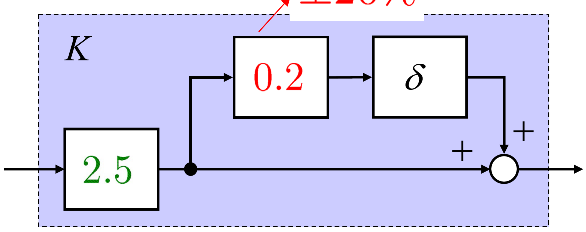
$$P(s) = \frac{K}{Ts+1} \qquad 2 \le K \le 3$$

$$K = 2.5(1+\delta \cdot 0.2), \quad |\delta| \le 1$$

$\pm 20\%$

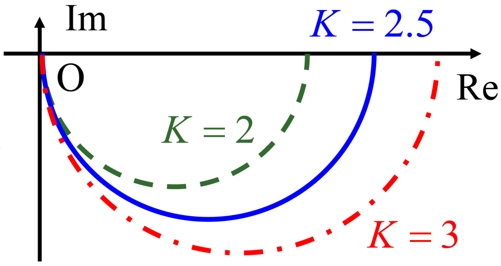
$\delta = 1 \Rightarrow K = 3$
$\delta = -1 \Rightarrow K = 2$





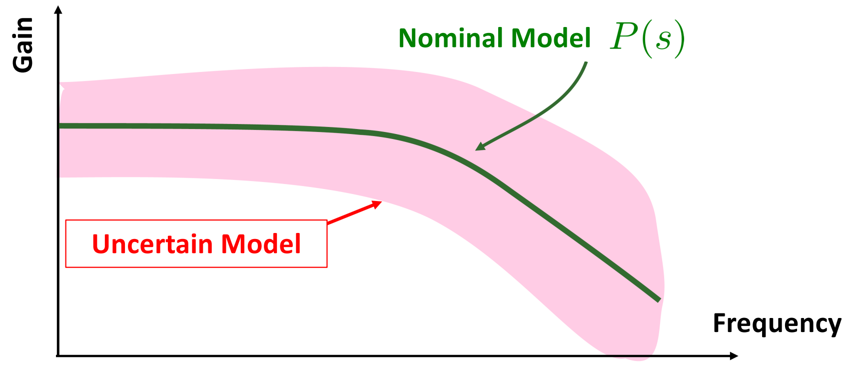
$T = 2.5; \quad K = 2, \ 2.5, \ 3$

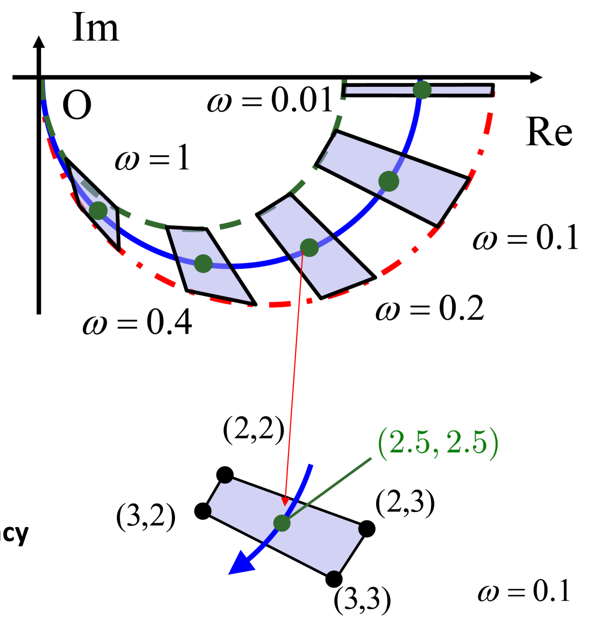
$$P : \left\{ \frac{2}{2.5s+1}, \frac{2.5}{2.5s+1}, \frac{3}{2.5s+1} \right\}$$



Uncertainty Modeling

$$P(s) = \frac{K}{Ts+1} \qquad 2 \le K \le 3 \qquad 2 \le T \le 3$$



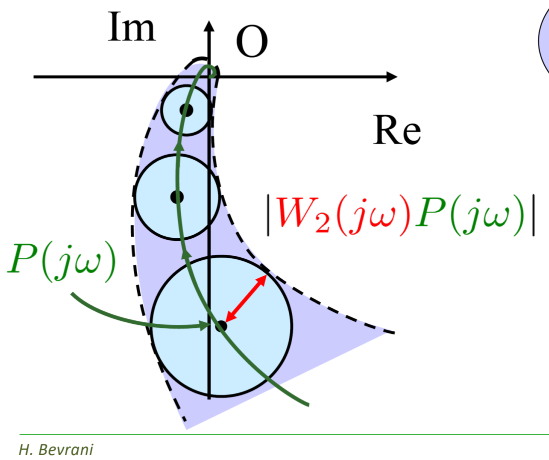


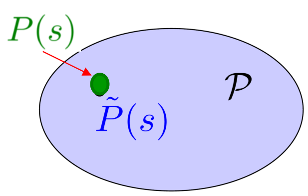
Uncertainty Modeling

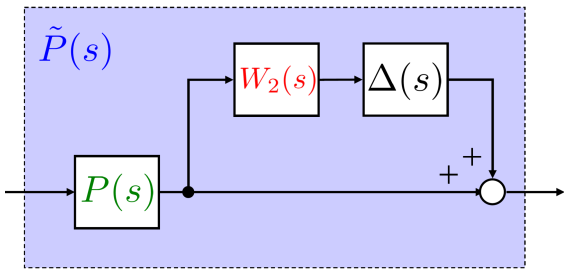
$$\tilde{P}(s) = (1 + \Delta(s) W_2(s)) P(s) \qquad |\Delta(j\omega)| \le 1, \forall \omega$$

Weighting function: Describe the "size" of uncertainty (Focus on gain)

$\tilde{P}(s) \in \mathcal{P}$







$$\frac{\tilde{P}}{P} - 1 = \frac{\tilde{P} - P}{P} = \Delta W_2$$

$$|\tilde{P} - P| = |\Delta W_2 P| \le |W_2 P|$$

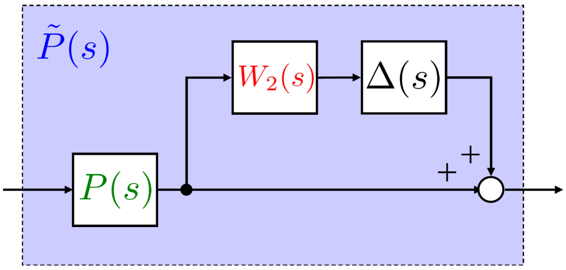
$$(\because |\Delta| \le 1)$$

Uncertainty Modeling

Output Multiplicative Uncertainty

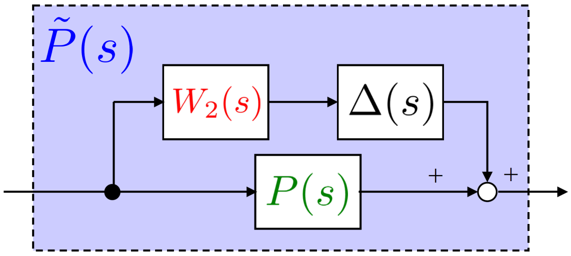
$$\tilde{P}(s) = (1 + \Delta(s)W_2(s))P(s)$$

$$\Rightarrow \frac{\tilde{P}}{P} - 1 = \frac{\tilde{P} - P}{P} = \Delta W_2$$



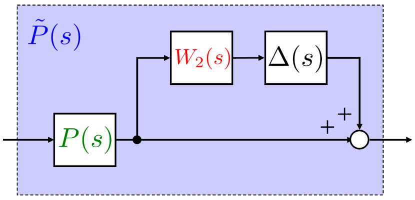
Additive Uncertainty

$$\tilde{P}(s) = P(s) + \Delta(s)W_2(s)$$



Uncertainty Modeling: Example

$$\tilde{P}(s) = \frac{1}{s+1}e^{-sL}, 0 \leq L \leq 1$$



$$\tilde{P}(s) = (1 + \Delta(s)W_2(s))P(s)$$

$$\Rightarrow \frac{\tilde{P}}{P} - 1 = \frac{\tilde{P} - P}{P} = \Delta W_2$$

Step 1: Nominal model $P(s) = \dfrac{1}{s+1}$

Step 2: Magnitude of multiplicative uncertainty (Upper limit)

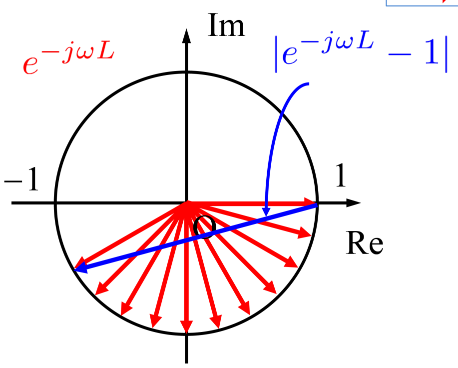
$$\left|\frac{\tilde{P}}{P} - 1\right| = \left|\frac{\tilde{P} - P}{P}\right| = |e^{-j\omega L} - 1|$$

$0 \leq \omega < \pi$:

$$|e^{-j\omega L} - 1| \leq |e^{-j\omega} - 1|$$

$\omega \geq \pi$: $(\because 0 \leq L \leq 1)$

$$|e^{-j\omega L} - 1|| \leq 2$$

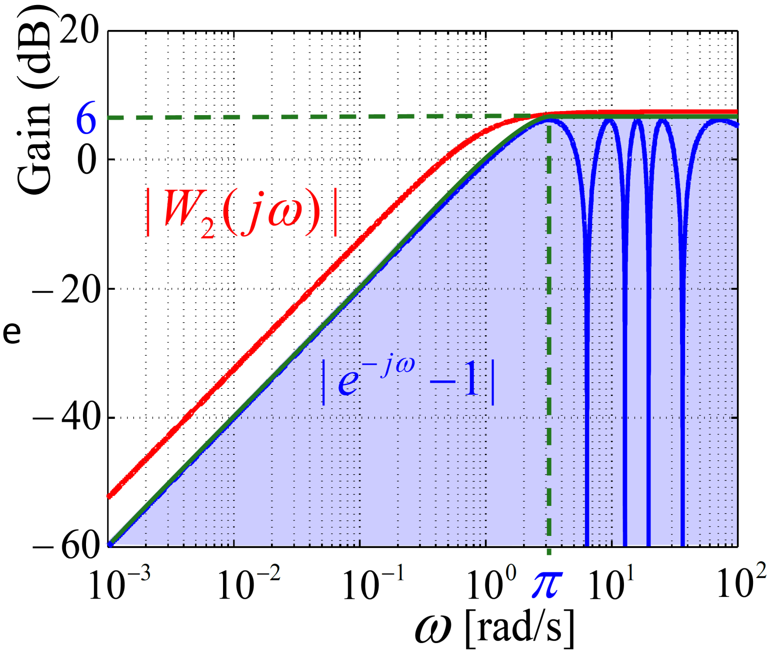


Uncertainty Modeling: Example

$0 \leq \omega < \pi: \quad |e^{-j\omega} - 1|$

$\omega \geq \pi: \quad 2$

Step 3: Find $W_2(s)$ to cover all possible uncertainties.



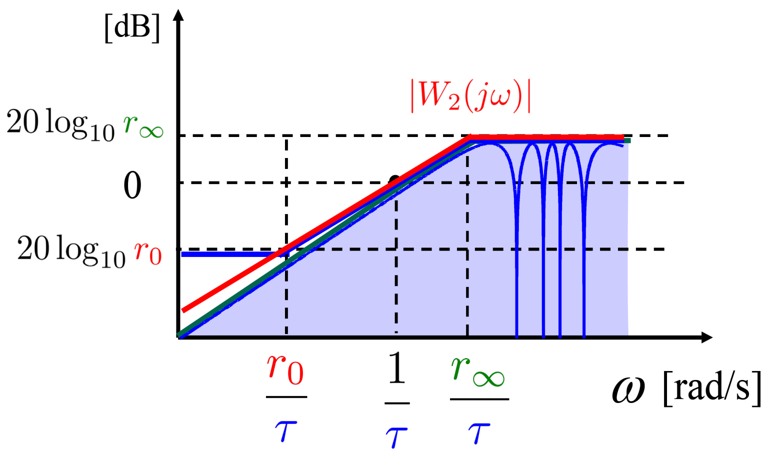
Uncertainty Weighting Function (Estimation Method)

$$W_2(s) = \frac{\tau s}{\frac{\tau}{r_\infty}s + 1} \qquad W_2(s) = \frac{\tau s + r_0}{\frac{\tau}{r_\infty}s + 1}$$

$\frac{1}{\tau}$: Frequency when uncertainty becomes 1

r_∞ : Uncertainty magnitude in high frequency band

r_0 : Uncertainty magnitude in low frequency band



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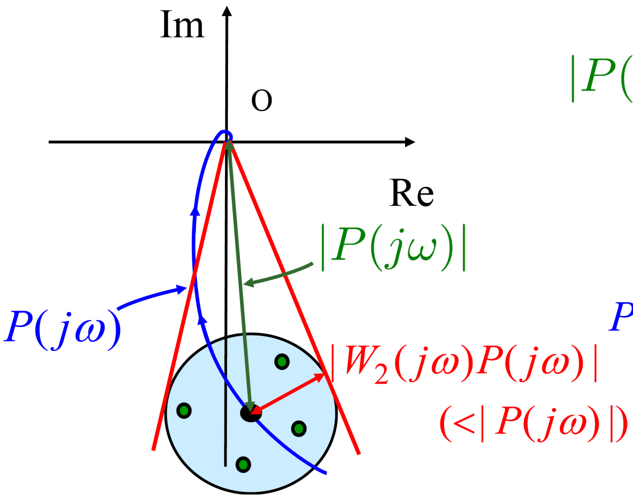
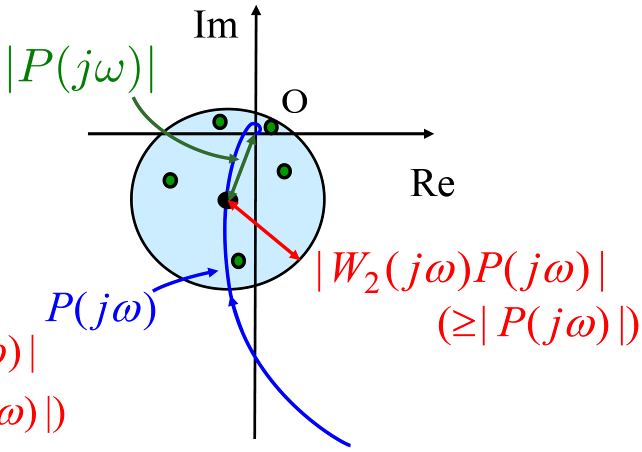


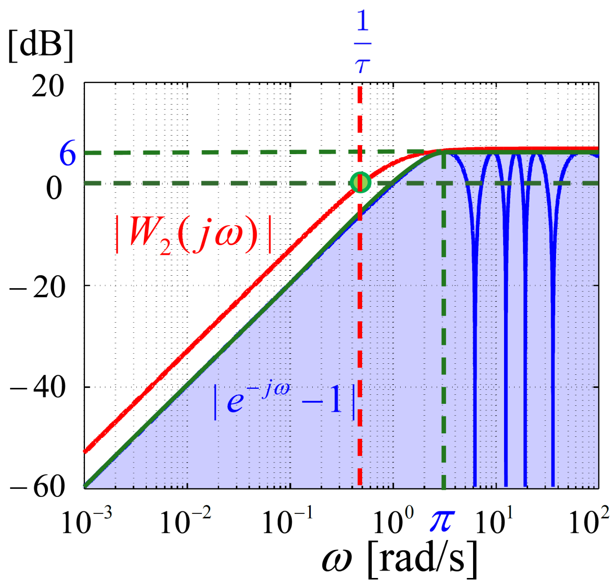
The magnitude of uncertainty less than 1



The magnitude of uncertainty more than 1

Continue

	STEP 3		STEP 2
$\frac{1}{\tau}$	0.48 rad/s $= 1/2.1$	$<$	1 rad/s
Uncertainty magnitude	-6dB=1/2		0dB=1
r_∞	6.4 dB $= 2.1$	$>$	6 dB $= 2$
r_0	$= 0$	$\geq$	0



- Nominal model $P(s) = \frac{1}{s+1}$

$$W_2(s) = \frac{2.1s}{s+1}$$

$$\mathcal{P} = \{(1+\Delta(s)W_2(s))P(s), |\Delta(j\omega)| \leq 1, {}^\forall\omega\}$$

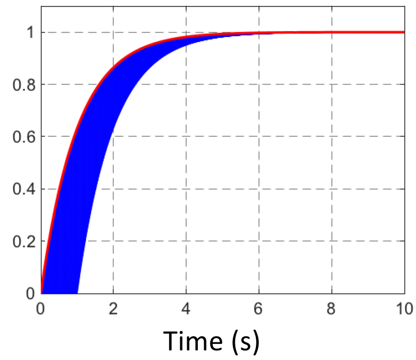
$$\tilde{P}(s) = \frac{1}{s+1}e^{-sL}, 0 \leq L \leq 1$$

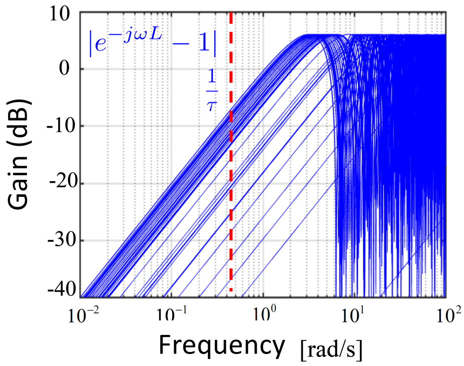
Continue

- Nominal model $P(s) = \dfrac{1}{s+1}$

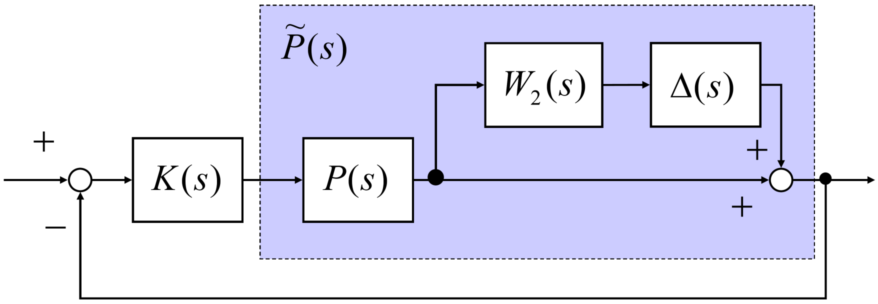
$$W_2(s) = \frac{2.1s}{s+1} \Rightarrow \mathcal{P} = \{(1+\Delta(s)W_2(s))P(s), |\Delta(j\omega)| \leq 1, {}^{\forall}\omega\}$$

$$\tilde{P}(s) = \frac{1}{s+1}e^{-sL}, 0 \leq L \leq 1$$





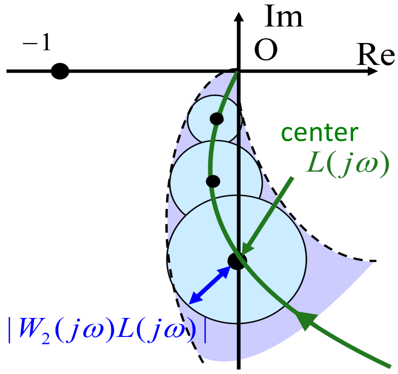
Robust Stability



$$\tilde{P}(s) = (1+\Delta(s)W_2(s))P(s)$$

$$\tilde{L}(s) = \tilde{P}(s)K(s) = (1+\Delta(s)W_2(s))P(s)K(s)$$

$$= L(s) + \Delta(s)W_2(s)L(s) \quad r(s)$$



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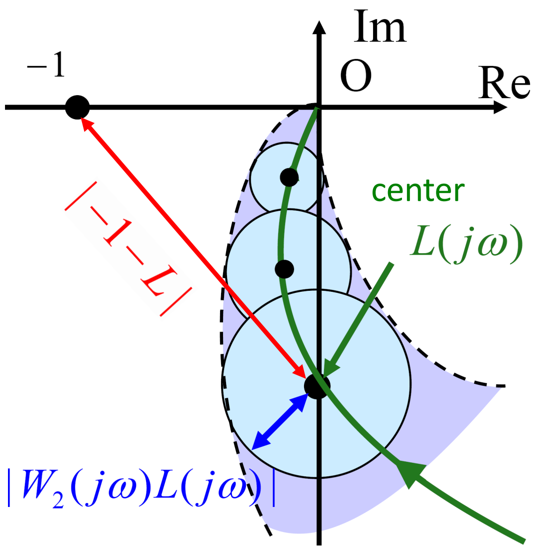
$|\tilde{L}-L| = |\Delta W_2 L| \le |W_2 L|$

$|-1-L| = |1+L|$

$|W_2 L| \ge |1+L|$: Instability Condition

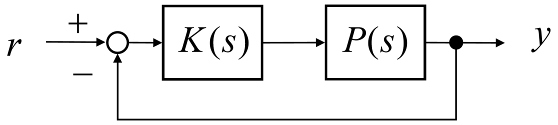
$|W_2 L| < |1+L|, \ \forall \omega$: Stability Condition

Robust Stability $\therefore \left| \dfrac{W_2 L}{1+L} \right| < 1, \ \forall \omega$



Complementary Sensitivity Function

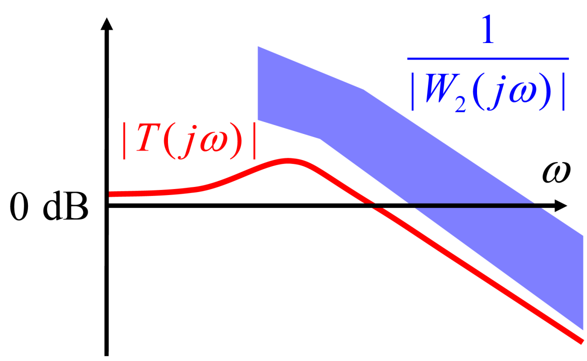
$$T(s) = \frac{P(s)K(s)}{1+P(s)K(s)} = \frac{L(s)}{1+L(s)}$$



- **Sensitivity Function:** $S(s) = \dfrac{1}{1+P(s)K(s)}$

$$S(s)+T(s) = \frac{1}{1+PK} + \frac{PK}{1+PK} = 1$$

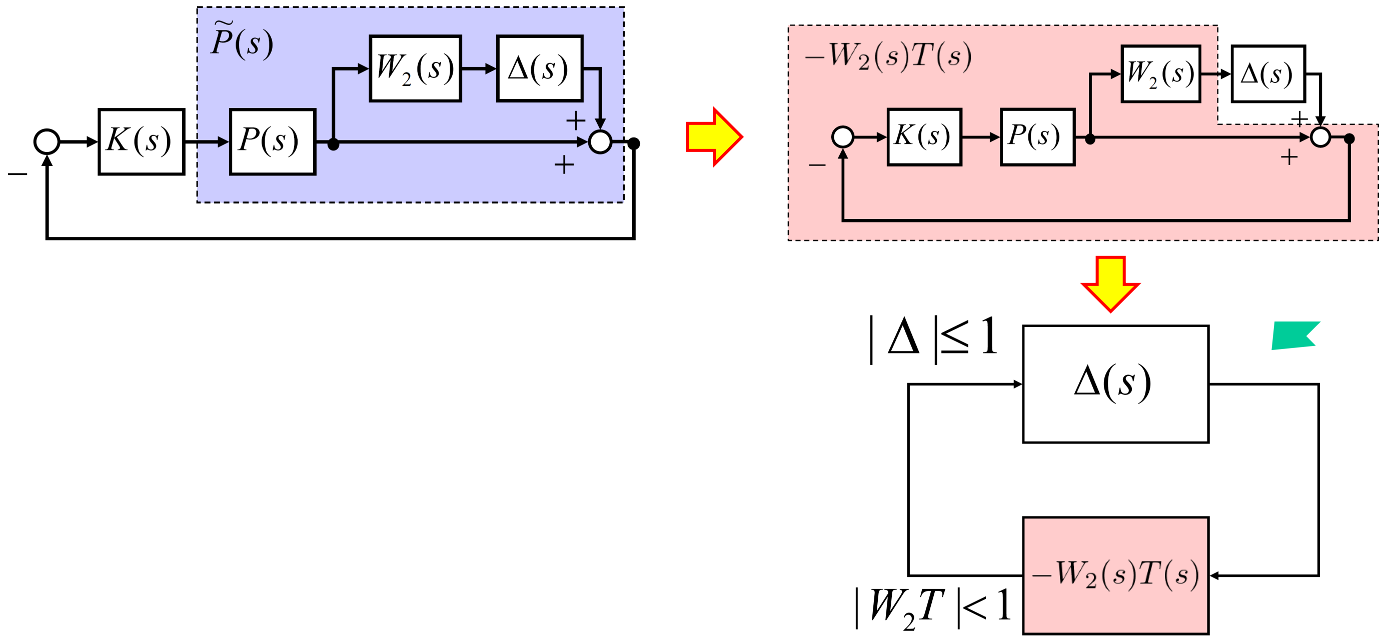
$$\left| \frac{W_2 L}{1+L} \right| < 1, \ \forall \omega \ \Rightarrow \ |W_2 T| < 1, \ \forall \omega$$



Robust Stability: $|T| < \dfrac{1}{|W_2|}, \ \forall \omega$

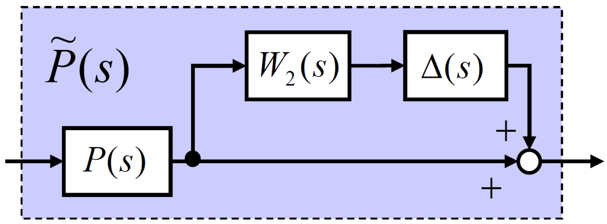
Smaller T is better at higher W2 frequencies

Robust Stability Framework



Example

$$\tilde{P}(s) = \frac{1}{s}e^{-sL}, \quad 0 \le L \le 0.1$$

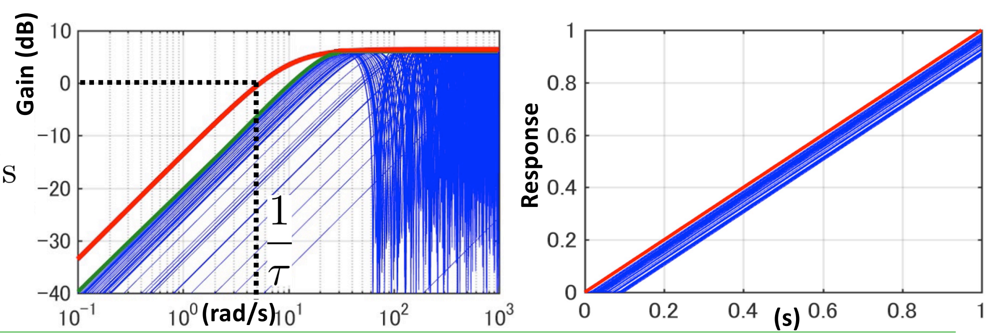


Nominal model: $P(s) = \frac{1}{s}$

Plants set: $\mathcal{P} = \{(1+\Delta(s)W_2(s))P(s), \; |\Delta(j\omega)| \le 1, \forall\omega\}$

$$W_2(s) = \frac{2.1s}{s+10}$$

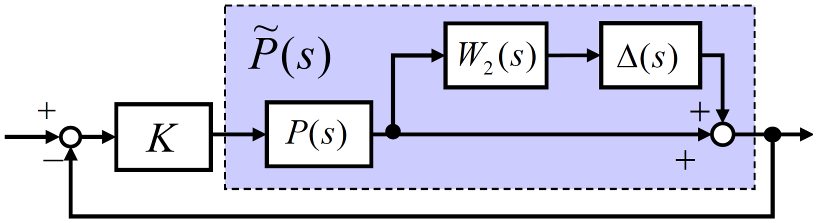
$$\tau = 0.21 \quad \frac{1}{\tau} = 4.8 \text{ rad/s}$$



Continue

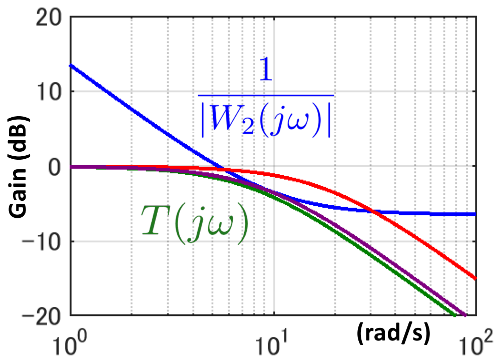
$$P(s) = \frac{1}{s} \qquad T(s) = \frac{K}{s+K}$$

$$W_2(s) = \frac{2.1s}{s+10}$$



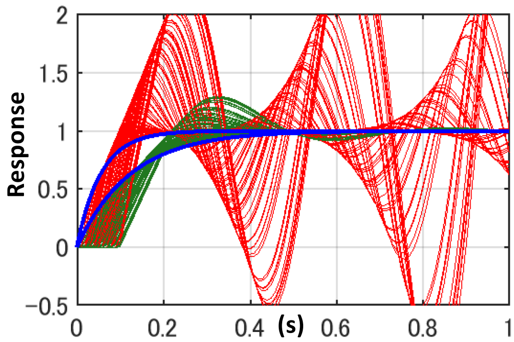
Robust Stability: $|T| < \frac{1}{|W_2|}, \ \forall\omega$

To satisfy robust stability: $0 < K < 9$



$K = 8$ ✔

$K = 18$ ✘



Project: Report 4

1) **In Report 2, you considered a deviation range for one or more parameters of your system. Model those deviation in output multiplicative uncertainty form.**
2) **Check the robust stability condition for the closed loop system with the designed P/PI controllers in Report 2.**

Deadline: The day before next Meeting

Please only use this email address: bevranih18@gmail.com

Thank You!