Robust Control Systems

Weighted Sensitivity and Performance

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Fall 2023

Contents

Reference

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2. M. Fujita, **Lecture Notes on Feedback Control Systems**, Tokyo Institute of Technology, 2019.

3. R. Smith, **Lecture Notes on Control Systems**, ETH Zurich, 2020.

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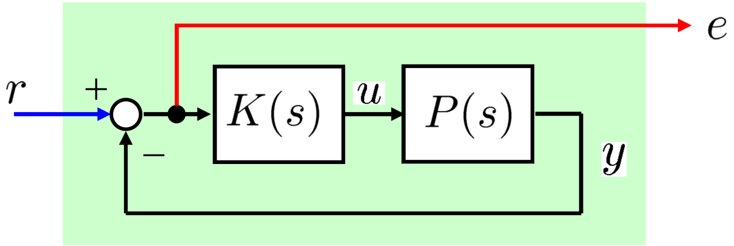
Sensitivity Optimization in Feedback Control

Feedback Performance = Sensitivity

H∞ norm: $$\min_{\text{Feedback } K} \|S\|_\infty = \min_{K} \|(I+PK)^{-1}\|_\infty$$ (System Gain)

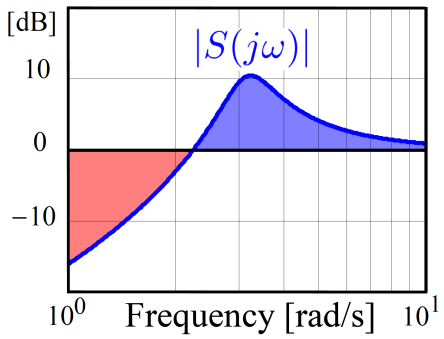


$$e = (I+PK)^{-1} r = Sr$$

Sensitivity from Reference to Error

Bode Sensitivity Integrals (Waterbed Effects) for Stable Plant

$$\int_0^\infty \log|\det S(j\omega)|d\omega = 0 \quad \begin{matrix}|\det S| < 1 \\ |\det S| > 1\end{matrix}$$



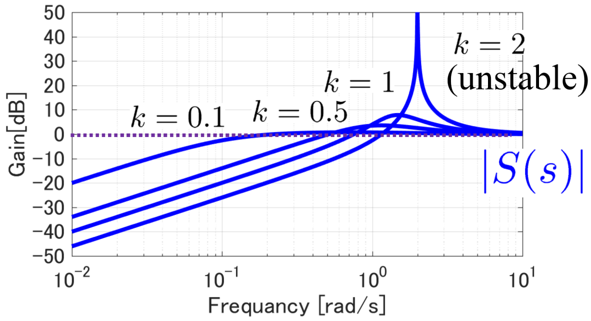
There exists a frequency range over which the magnitude of the sensitivity function exceeds 1 if it is to be kept below 1 at the other frequency range.

Example:

$$P(s) = \frac{2-s}{2+s}$$ RHP(Right-Half Plane) Zero

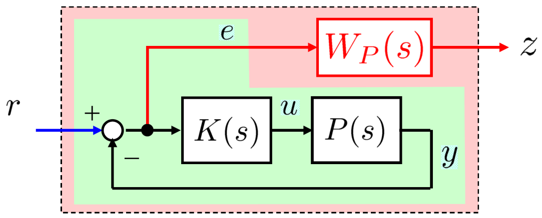
$$K(s) = \frac{k}{s}$$

$$S(s) = \frac{1}{1+P(s)K(s)}$$



(Ref 1, pp. 167, 170, 223)

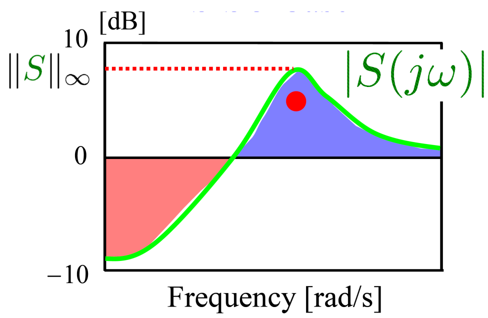
Weighted Sensitivity



$$e = (I+PK)^{-1}r = Sr$$

$\|S\|_\infty$ Small?

SISO case: Waterbed Effects



$\|W_P S\|_\infty$ Small! Intractable ⇒ Tractable!

Performance weight
transfer function matrix

$$W_P(s) = \begin{bmatrix} w_{p1}(s) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & w_{pn}(s) \end{bmatrix} \left(= \begin{bmatrix} w_p(s) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & w_p(s) \end{bmatrix} \right)$$

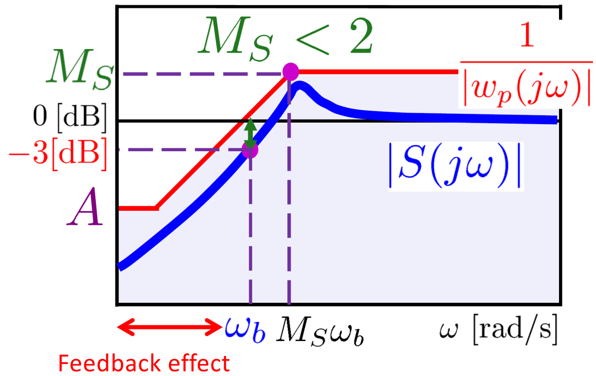
(Ref 1, p. 60)

Performance Weight

First-order Performance Weight

$$w_p(s) = \frac{\frac{1}{M_S}s + \omega_b}{s + \omega_b A}$$

$$w_p(s) = \frac{\left(s/M_S^{1/n} + \omega_b\right)^n}{\left(s + \omega_b A^{1/n}\right)^n}$$



ω_b : the frequency at which the asymptote of $1/|w_p(j\omega)|$ crosses 1, and the bandwidth requirement approximately

M_S: $1/|w_p(j\omega)|$ at high frequencies ($M_S < 2$: Rule of thumb)

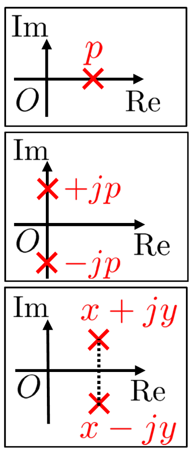
A : $1/|w_p(j\omega)|$ at low frequencies

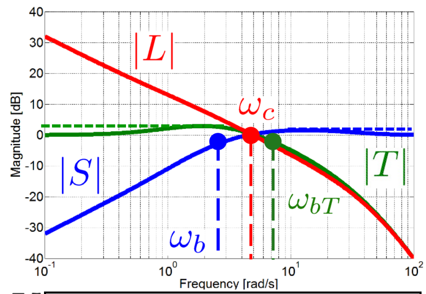
(Ref 1, pp. 62, 80)

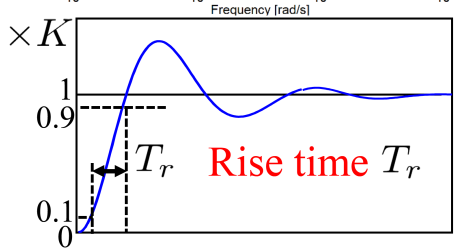
Stability and Performance

Unstable Plant:

- Real RHP Poles: $2p < \omega_c$
- Imaginary Poles: $1.15|p| < \omega_c$
- Complex RHP Poles: $0.67\left(x + \sqrt{4x^2 + 3y^2}\right) < \omega_c$







Stable Plant:

- **First-order System:**

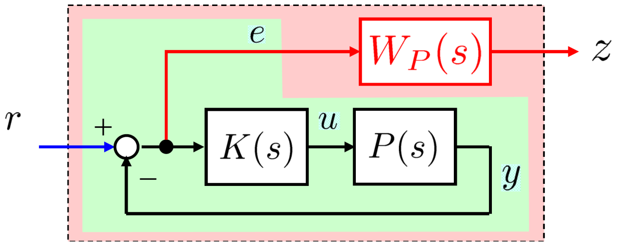
$$G_1(s) = \frac{K}{Ts+1} \quad K > 0 \quad T > 0$$

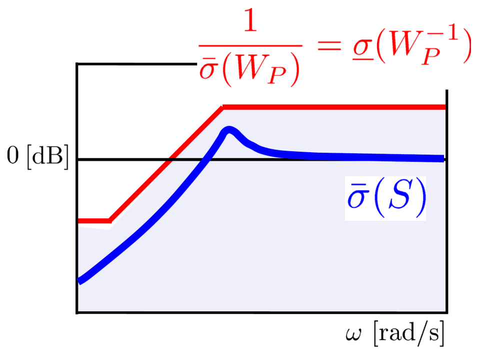
$$T_r = (\ln 9)T \approx 2.2T \qquad \frac{2.2}{T_r} \leq \frac{1}{T} \leq \omega_c$$

- **Second-order System:** $G_2(s) = \dfrac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ $\quad \omega_n > 0 \quad \zeta \geq 0$ $\qquad T_r = \dfrac{\pi/2 + \arcsin\zeta}{\omega_n\sqrt{1-\zeta^2}} \qquad \dfrac{1.8}{T_r} \leq \omega_r \leq \omega_c$

(Ref 1, Sec. 5.9)

Nominal Performance (NP)





$$\bar{\sigma}(S(j\omega)) < \frac{1}{\bar{\sigma}(W_P(j\omega))} \quad \forall\omega$$

$$\bar{\sigma}(W_P(j\omega)S(j\omega)) < 1 \quad \forall\omega$$

$$\underline{\sigma}(A)\bar{\sigma}(B) \le \bar{\sigma}(AB)$$
$$\bar{\sigma}(AB) \le \bar{\sigma}(A)\bar{\sigma}(B)$$
$$\bar{\sigma}(A^{-1}) = \frac{1}{\underline{\sigma}(A)}$$
$$\|A\|_\infty = \bar{\sigma}(A)$$

- Nominal Performance (NP) Test:

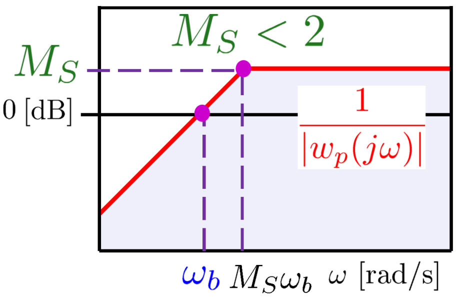
$$\|W_P(s)S(s)\|_\infty < 1$$

(Ref 1, p. 81)

Nominal Performance Test in SISO Systems

(NP) Test: $|S(j\omega)| < \frac{1}{|w_p(j\omega)|} \quad \forall\omega$

Example: $S = \frac{s^2 + s}{s^2 + 0.7s + 0.07}$ $\quad w_p(s) = \frac{\frac{1}{M_S}s + \omega_b}{s}$



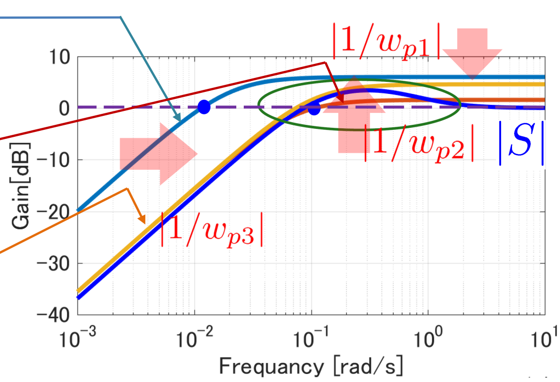
1) w_{p1} $\omega_b = 0.01, M_S = 2$ NP ✓

⇓ ω_b : fast M_S: small

2) w_{p2} $\omega_b = 0.06, M_S = 1.2$ NP ✗

⇓ M_S: large

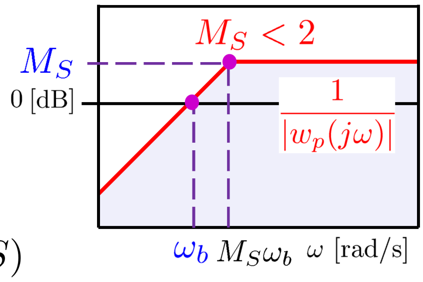
3) w_{p3} $\omega_b = 0.06, M_S = 1.7$ NP ✓



(Ref 1, p. 60)

Nominal Performance Test in SISO Systems

(NP) Test: $\|W_P(s)S(s)\|_\infty < 1$



Example: $S = \dfrac{s^2+s}{s^2+0.7s+0.07}$ $\quad w_p(s) = \dfrac{\frac{1}{M_S}s+\omega_b}{s}$

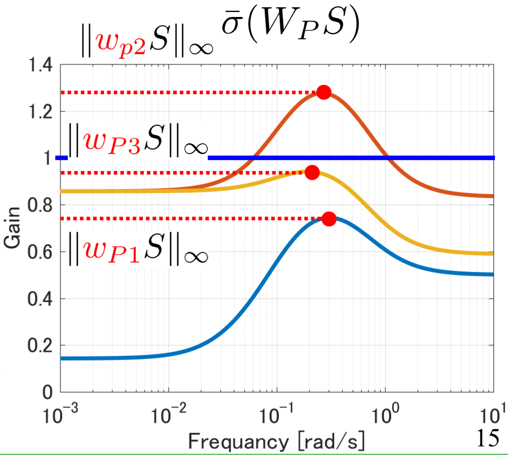
1) w_{p1} $\omega_b = 0.01, M_S = 2$ **NP ✓**

ω_b : fast M_S: small

2) w_{p2} $\omega_b = 0.06, M_S = 1.2$ **NP ✗**

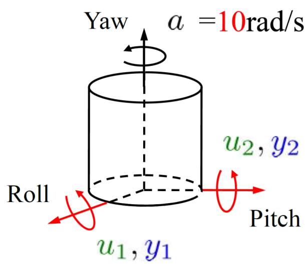
M_S: large

3) w_{p3} $\omega_b = 0.06, M_S = 1.7$ **NP ✓**



Example: Spinning Satellite

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \begin{bmatrix} \frac{s-100}{s^2+100} & \frac{10s+10}{s^2+100} \\ \frac{-10s-10}{s^2+100} & \frac{s-100}{s^2+100} \end{bmatrix}$$

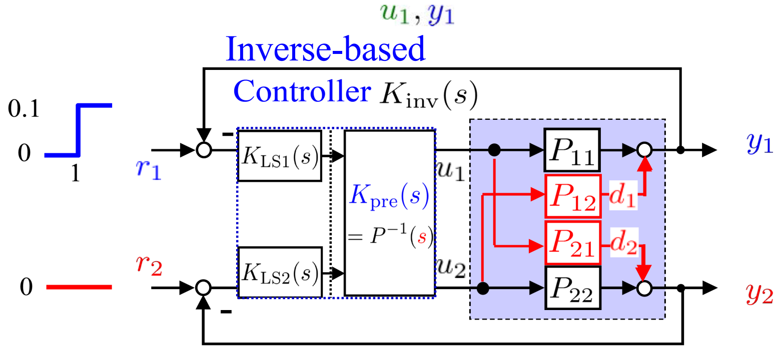


Inverse-based Controller $K_{\text{inv}}(s)$ **:**

$$P^{-1}(s) = 0.01\begin{bmatrix} s-100 & -10s-10 \\ 10s+10 & s-100 \end{bmatrix}$$

$$P'(s) = P(s)P^{-1}(s) = I$$

$$K_{\text{inv}}(s) = P^{-1}(s)\begin{bmatrix} K_{\text{LS1}}(s) & 0 \\ 0 & K_{\text{LS2}}(s) \end{bmatrix}$$



Loop Shaping Design: $K_{\text{LS1}}(s) = K_{\text{LS2}}(s) = \dfrac{900}{s(s+30)}$ Desired Loop ⇒ $L_{\text{target}}(s) = \dfrac{900}{s(s+30)}I_2$

Example: Spinning Satellite

Performance Weight

$$W_P(s) = \begin{bmatrix} w_p(s) & 0 \\ 0 & w_p(s) \end{bmatrix} \left(= \begin{bmatrix} w_{p1} & 0 \\ 0 & w_{p2} \end{bmatrix} \right) \qquad w_p(s) = \frac{\frac{1}{M_s}s + \omega_b}{s + \omega_b A}$$

MATLAB Command

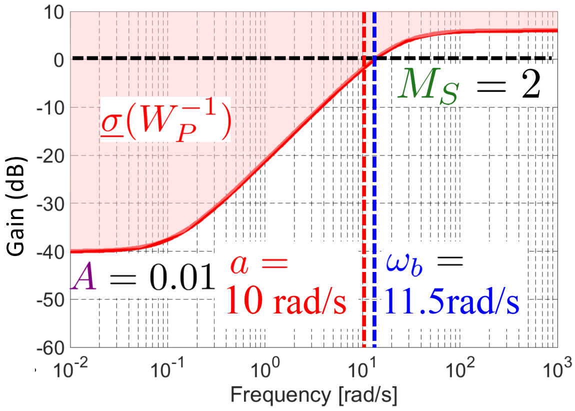
```
Ms = 2; A = 1e-2; wb = 11.5;
wP = tf([1/Ms wb], [1 wb*A]);
WP = eye(2)*wP;
figure
sigma(WP)
hold on; grid on;
```

- Poles on the imaginary axis: $p = \pm aj = \pm 10\, j$
- Gain crossover frequency:

  $\omega_c > 1.15|p| = 11.5 \text{ rad/s} = \omega_b$
- Phase stabilization:

  $\omega_c <$ System bandwidth of Actuator/Sensor/Controller

  $M_S \leq 2 \longrightarrow M_S = 2$
- Steady state error: $e \leq 0.01 \longrightarrow A = 0.01$



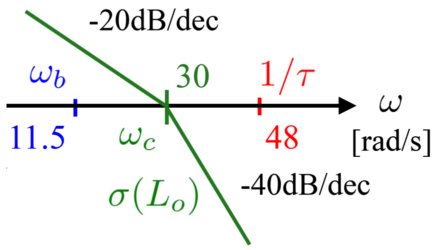
Example: Spinning Satellite

Plant:

$$P(s) = \begin{bmatrix} \frac{s-100}{s^2+100} & \frac{10s+10}{s^2+100} \\ \frac{-10s-10}{s^2+100} & \frac{s-100}{s^2+100} \end{bmatrix}$$

Controller (Inverse-based Controller):

$$K_{\text{inv}}(s) = P^{-1}(s) \begin{bmatrix} \frac{900}{s(s+30)} & 0 \\ 0 & \frac{900}{s(s+30)} \end{bmatrix}$$



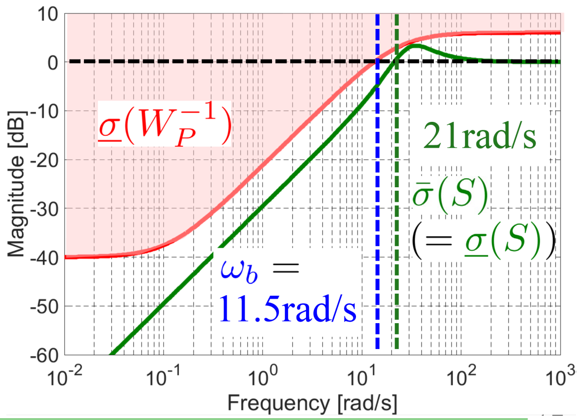
Target Loop Transfer Function: $L(s) = PK_{\text{inv}} = \dfrac{900}{s(s+30)} I_2$

(Output) Sensitivity Function: $S(s) = (I + PK_{\text{inv}})^{-1}$

$$\|W_P S\|_\infty = 0.8935 < 1$$ **NP** ✓

MATLAB Command

```
KI = inv(Pnom)*tf([1],[1 30 0])*diag([900 900]);
FI = loopsens(Pnom,KI);
sigma(FI.So) ;
hinfSo = normhinf(WP*FI.So)
```



Optimal Sensitivity Minimization Problem

Find a stabilizing controller K which make smaller (minimize)

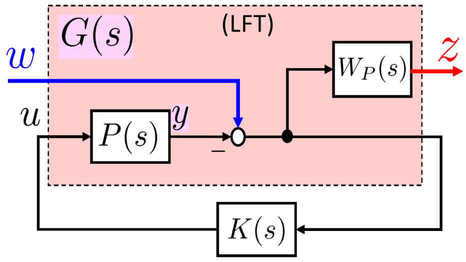
$$\|W_P(s)S(s)\|_\infty$$

Intractable

Sensitivity Minimization Problem (H∞ Control)

Given $\gamma > \gamma_{min}$, find all stabilizing controllers K such that

$$\|W_P(s)S(s)\|_\infty < \gamma$$



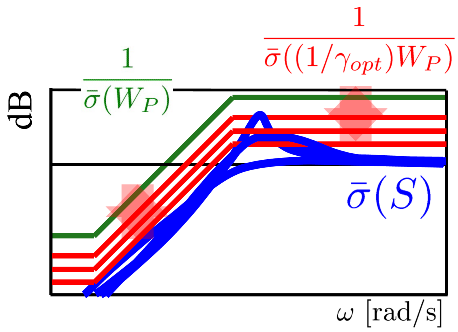
1) $\|W_PS\|_\infty < \gamma_1$ ✓ $^\exists K_1$

2) $\|W_PS\|_\infty < \gamma_2$ ✗ no K_2

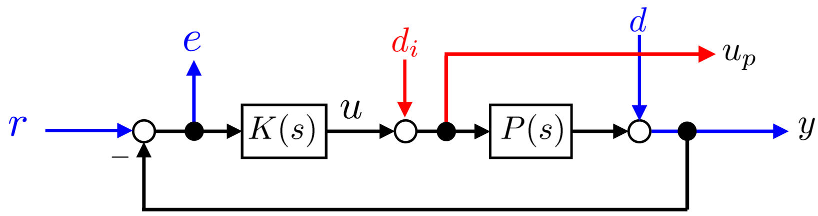
3) $\|W_PS\|_\infty < \gamma_3$ ✓ $^\exists K_3$

⋮

$\frac{1}{\gamma_{opt}} W_P$



Sensitivity for MIMO Systems



- Sensitivity to Output Disturbance d :

Output Sensitivity Function: $S_o(s) = (I + P(s)K(s))^{-1}$

- Sensitivity to Input Disturbance di :

Input Sensitivity Function: $S_i(s) = (I + K(s)P(s))^{-1}$

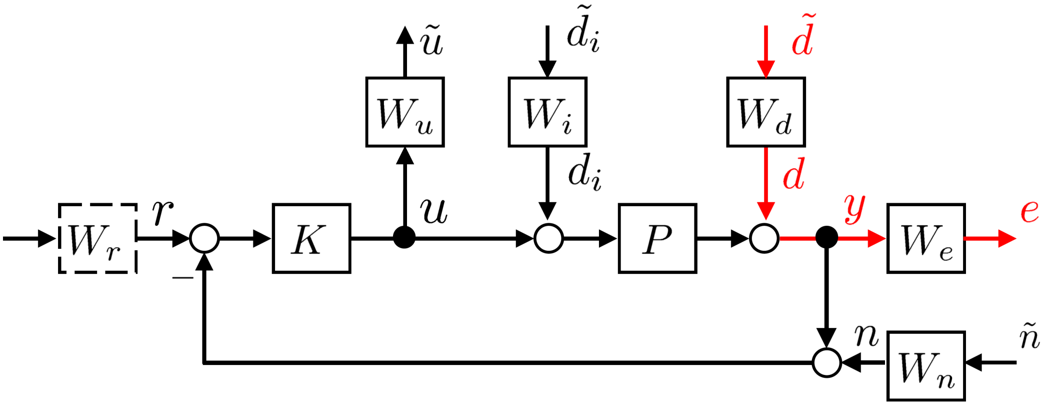
For SISO Systems: $S_i = S_o$

but for MIMO Systems: $PK \neq KP \rightarrow S_i \neq S_o$

Good disturbance rejection at output does not always mean good rejection at input

(Ref 1, p. 70)

Standard Feedback Configuration with Weights



○ Sensitivity Minimization Problem

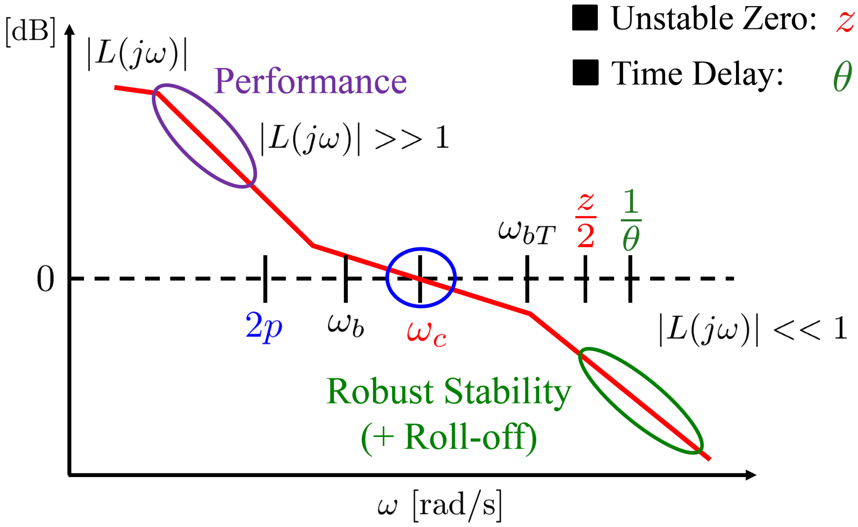
$$\text{find } K(s) \text{ s.t. } \|W_e(s)S_o(s)W_d(s)\|_\infty < \gamma$$

(Ref 1, p. 363)

SISO Loop Shaping

Loop Shaping: Gives us graphical interpretation (Bode Plot, and System Gain)

- Unstable Pole: p
- Unstable Zero: z
- Time Delay: θ



(Ref 1, pp. 41, 42, 343)

RHP Poles/Zeros, Time Delays and Sensitivity (SISO Systems)

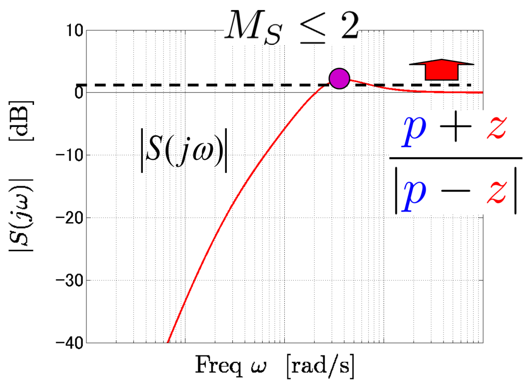
For systems with a RHP pole p and RHP zero z (or a time delay θ), any stabilizing controller gives sensitivity functions with the property

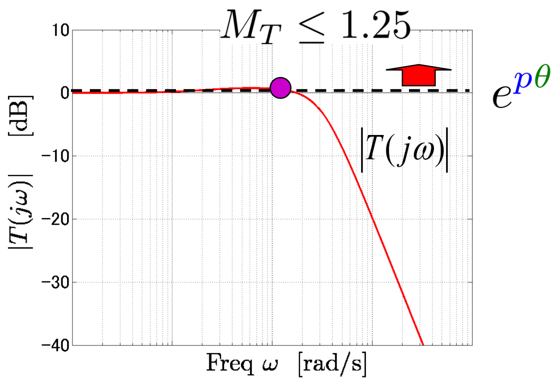
$$M_S = \sup_{\omega} |S(j\omega)| \geq \frac{|p+z|}{|p-z|}$$

The zero and the pole must be sufficiently far apart

$$M_T = \sup_{\omega} |T(j\omega)| \geq e^{p\theta}$$

The product of RHP pole and time delay must be sufficiently small





Norms

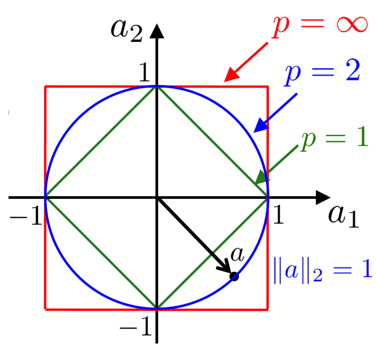
○ **Key properties:**

1. Non-negative $\quad \|e\| \geq 0$
2. Positive $\quad \|e\| = 0 \text{ iff } e = 0$
3. Homogeneous $\quad \|\alpha e\| = |\alpha| \|e\|, \ \forall \alpha$:scalar
4. Triangle inequality $\quad \|e_1 + e_2\| \geq \|e_1\| + \|e_2\|$

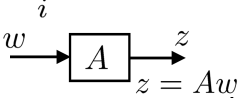
○ **Vector Norm (Ex.)**

$$\|a\|_2 = \sqrt{\sum_i |a_i|^2}$$ (Euclidean Vector Norm)

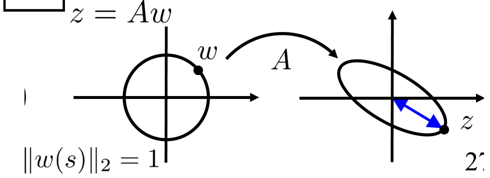
$$\|a\|_1 = \sum_i |a_i|, \quad \|a\|_\infty = \max_i |a_i|$$



○ **(Induced) Matrix Norm**

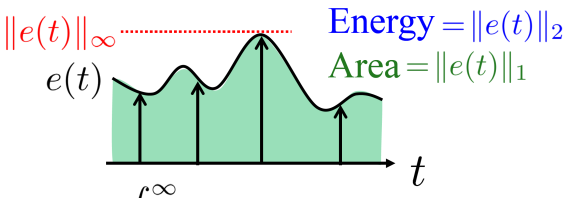


Ex.: $\|A\|_{i2} = \max_{\omega \neq 0} \frac{\|z\|_2}{\|\omega\|_2} = \bar{\sigma}(A)$



Norms

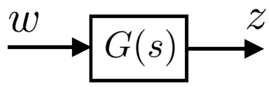
- **Signal Norm**



Examples:

- "Energy of signal" ($\mathcal{L}_2$-norm, $\mathcal{L}$: Lebesgue space) $\|e(t)\|_2^2 = \int_{-\infty}^{\infty} \sum_i |e_i(t)|^2 dt$
- Integral absolute error $\|e(t)\|_1 = \int_{-\infty}^{\infty} \sum_i |e_i(t)| dt$
- "maximum value over time" $\|e(t)\|_\infty = \max_t \left(\max_i |e_i(t)| \right)$

System Norm (MIMO): $\|G(s)\|_\infty = \max_{\omega \neq 0} \dfrac{\|z\|_2}{\|\omega\|_2} = \max_\omega \bar{\sigma}(G(j\omega))$ (System Gain)



$$\|G(s)\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}(G(j\omega)^H G(j\omega))d\omega} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_i \sigma_i^2(G(j\omega))d\omega}$$

(Ref 1, A.5)

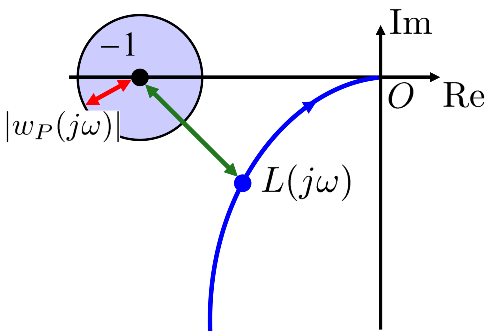
Nominal Performance in SISO Systems

$$|w_P S| < 1 \ \forall\omega \iff |w_P| < |1+L| \ \forall\omega$$

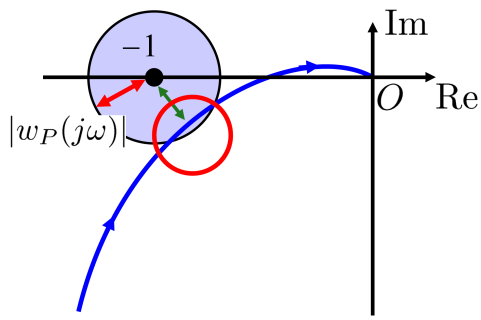
$$S = \frac{1}{1+PK} = \frac{1}{1+L}$$

- **Nyquist Plot**

L should be away from $(-1,0)$ by $|w_P|$



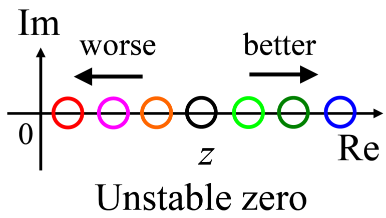
$|w_P| < |1+L| \ \forall\omega$ ✔

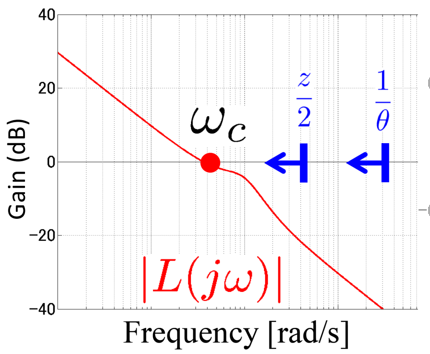


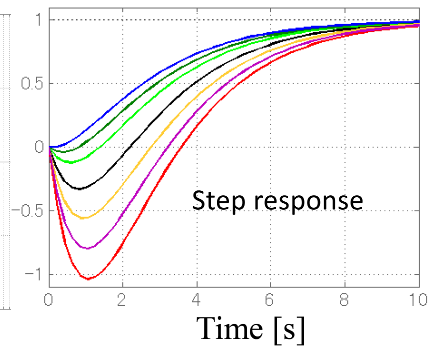
$|w_P| > |1+L| \ \exists\omega$ ✘

Fundamental Limitations

- Bound on the Crossover Frequency ω_c
- RHP (Right half-plane) Zero z $\omega_c < \frac{z}{2}$
- Fast RHP Zeros(z large): Loose Restrictions
- Slow RHP Zeros(z small): Tight Restrictions
- Time delay θ: $\omega_c < \frac{1}{\theta}$



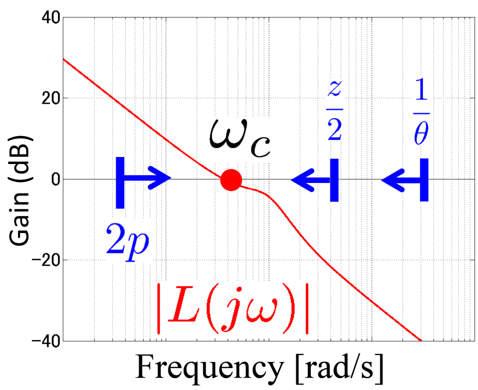




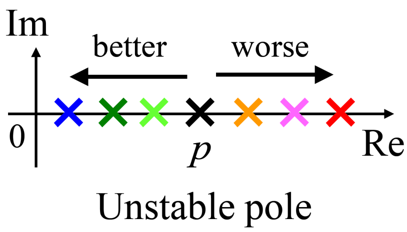
(Ref 1, p. 183)

Fundamental Limitations

- Bound on the Crossover Frequency ω_c
- RHP (Right half-plane) Poles p $\omega_c > 2p$
- Slow RHP Poles (p small): Loose Restrictions
- Fast RHP Poles (p large): Tight Restrictions



Poles on imaginary axis: $\pm pj$ $\omega_c > 1.15|p|$



(Ref 1, pp. 192, 194)

Fundamental Limitations in MIMO Systems

- **Algebraic Constraint**

$$S + T = I \Rightarrow \begin{array}{l} |\bar{\sigma}(S) - 1| \le \bar{\sigma}(T) \le \bar{\sigma}(S) + 1 \\ |\bar{\sigma}(T) - 1| \le \bar{\sigma}(S) \le \bar{\sigma}(T) + 1 \end{array}$$

$$\Downarrow$$

$$|\bar{\sigma}(S) - \bar{\sigma}(T)| \le 1$$

$\bar{\sigma}(S)$ is large if and only if $\bar{\sigma}(T)$ is large

- **Bounds on Peaks**

$$M_{S,\min} \triangleq \min_K \|S\|_\infty \ , \ M_{T,\min} \triangleq \min_K \|T\|_\infty$$

(Ref 1, Sec. 6.2)

Thank You!