Robust Control Systems

Uncertainty and Robust Performance

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Contents

Reference

1. S. Skogestadand I. Postlethwaite, **Multivariable Feedback Control; Analysis and Design**, Second Edition, Wiley, 2005.

2. M. Fujita, **Lecture Notes on Feedback Control Systems**, Tokyo Institute of Technology, 2019.

3. R. Smith, **Lecture Notes on Control Systems**, ETH Zurich, 2020.

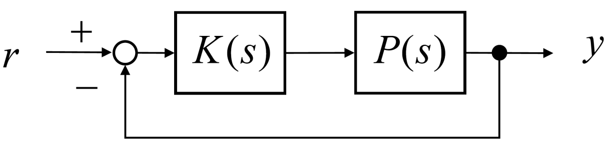
4. H. Bevrani, **Lecture Notes on Robust Control**, University of Kurdistan, 2018.

5. M. Hirata, **Practical Robust Control**, CORONA Press , 2017 (In Japanese).

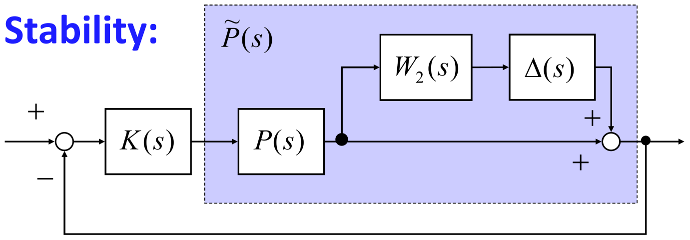
Nominal Stability and Robust Stability (Review)

○ **Nominal Stability:** $T(s) = \dfrac{P(s)K(s)}{1+P(s)K(s)} = \dfrac{L(s)}{1+L(s)}$

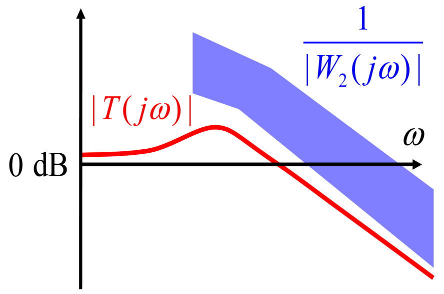
$\phi = D_P D_K + N_P N_K = 0$ ⇨ $(S, T, KS, PS$ Stable)



○ **Robust Stability:**

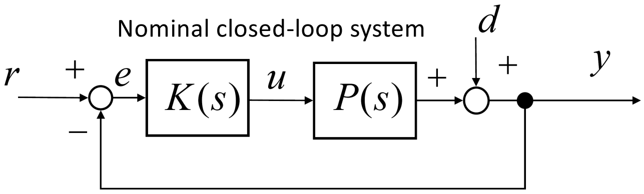


$\left|\dfrac{W_2 L}{1+L}\right| < 1, \ ^\forall\omega \Rightarrow |W_2 T| < 1, \ ^\forall\omega$ ⇨ $|T| < \dfrac{1}{|W_2|}, \ ^\forall\omega$



Nominal Performance

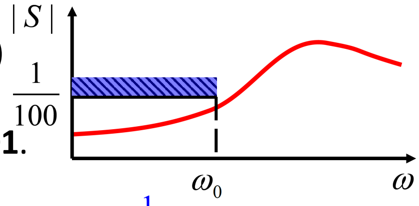
Sensitivity to parameter changes: $\Delta_T = \frac{1}{1+PK}\Delta_P$



Disturbance sensitivity: $y = \frac{1}{1+PK}d$

Tracking: $e = \frac{1}{1+PK}r$

Feedback performance index $S = \frac{1}{1+PK}$ (Smaller is better: $y = Sd$)

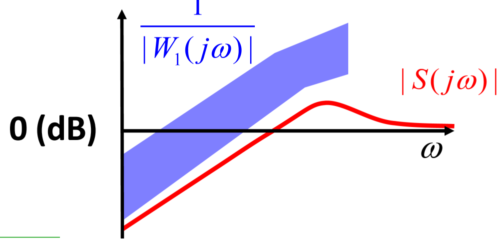


Example: Attenuated disturbances in low frequencies less than **0.01**.

$$|S| < \frac{1}{100} \qquad \forall \omega \le \omega_0$$

$$|W_1| \ge 100, \qquad \forall \omega \le \omega_0$$ (W1: Weight function)

$|S| < \frac{1}{|W_1|}, \ \forall \omega$ ⇨ $|W_1 S| < 1, \ \forall \omega$ Nominal Performance

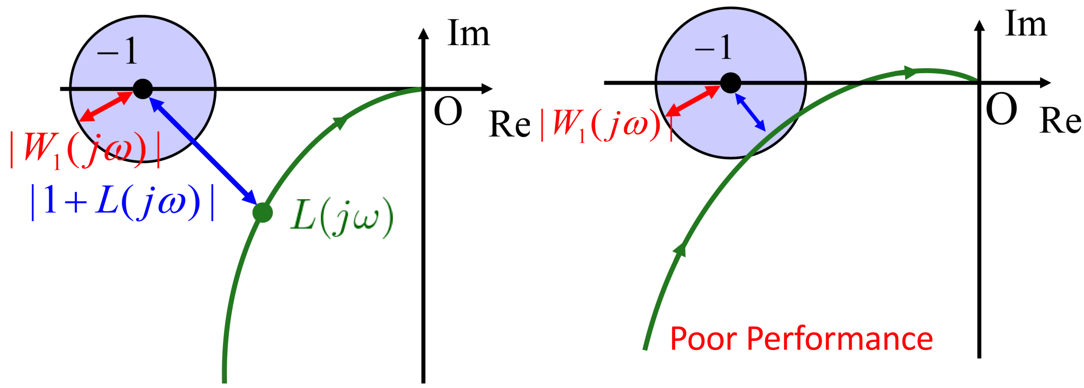


Nominal Performance

Nominal Performance: $|W_1 S| < 1, \ \forall \omega$ ⇨ $|W_1| < |1+L|, \ \forall \omega$

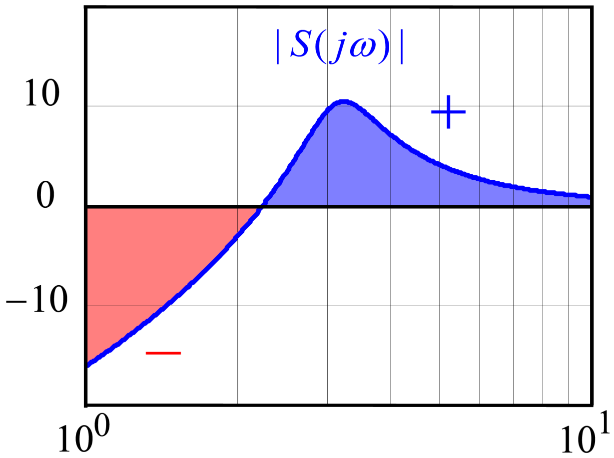
$$S = \frac{1}{1+PK} = \frac{1}{1+L}$$

Nominal performance by vector trajectory



Nominal Performance

The sensitivity function S cannot be reduced in all frequency bands (*Water bed effect*).



Bode Sensitivity Integral:

$$\int_0^\infty \log|S(j\omega)|\,d\omega = 0$$

$$|S|<1 \qquad (\log|S|<0)$$

In another frequency band

$$|S|>1 \qquad (\log|S|>0)$$

This fact makes a limit for more Performance improvement.

$$S+T=1 \qquad \left(S=\frac{1}{1+PK},\quad T=\frac{PK}{1+PK}\right)$$

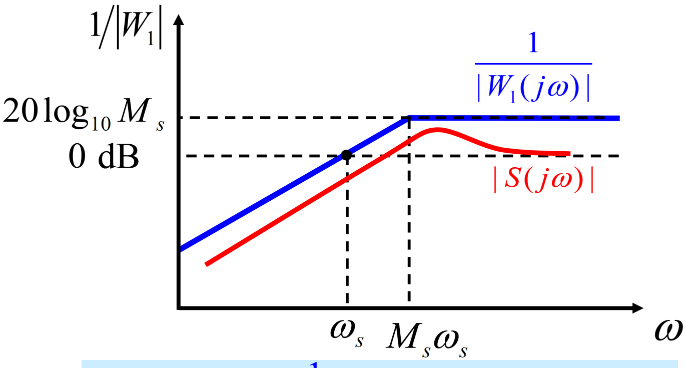
Nominal Performance

Performance weighting function $W_1(s)$

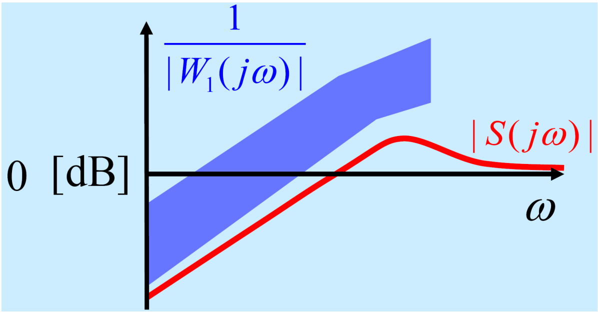
$$|S|<\frac{1}{|W_1|},\ \forall\omega$$

$$\Rightarrow |S(j\omega)|<\left|\frac{j\omega}{j\omega/M_s+\omega_s}\right|$$

$$W_1(s)=\frac{s/M_s+\omega_s}{s}$$



Sensitivity function and nominal performance



Nominal Performance

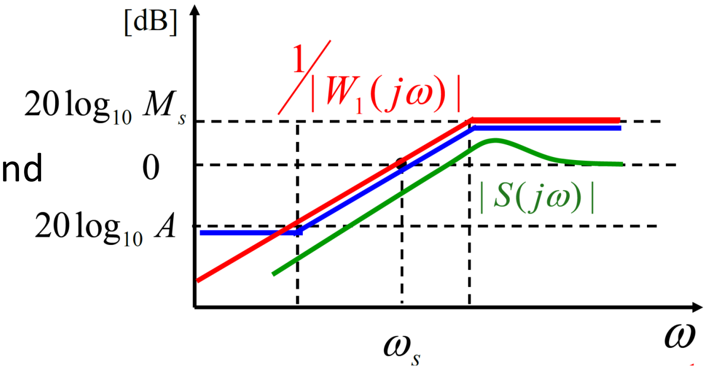
Performance Weighting function $W_1(s)$

$$W_1 = \frac{\frac{1}{M_s}s + \omega_s}{s} \qquad W_1 = \frac{\frac{1}{M_s}s + \omega_s}{s + \omega_s A}$$

ω_s : Frequency when the size of the sensitivity function becomes 1

M_s : High frequency performance
(a suggestion: $M_s \leq 2$)

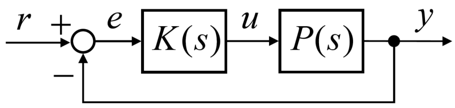
A : Performance in the low frequency band



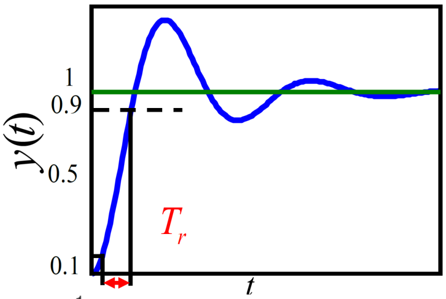
Example

Find W1 and K for approaching a 2-time faster Rice time (Assume: $T_r \approx 2.2T$ [s]).

$$P(s) = \frac{1}{s} \quad K(s) = K \quad G_{yr} = \frac{K}{s+K}$$



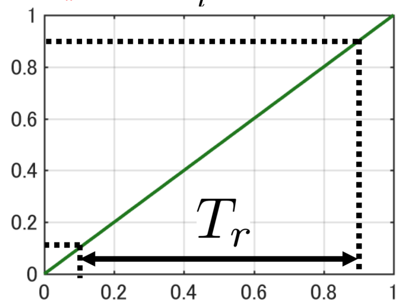
Step response of nominal model: $T_r = 0.8$ s



Objective: ⇨ $T_r \leq 0.4$ s (Desired Rise time)

○ Find W_1: $W_1 = \frac{\frac{1}{M_s}s + \omega_s}{s}$

$$T_r \cong 2.2T \leq 0.4 \text{ s} \quad ⇨ \quad \frac{2.2}{0.4} = 5.5 \leq \frac{1}{T} \leq \omega_s$$

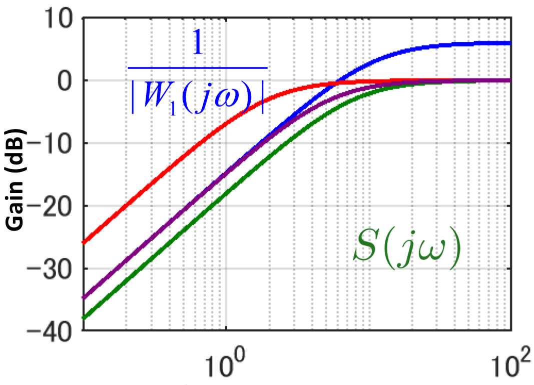


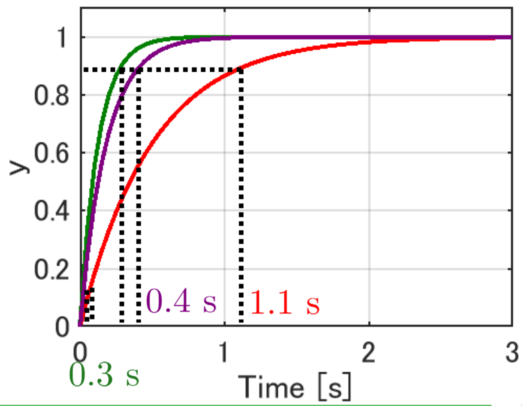
Continue

$\omega_s = 5.5$
$M_s = 2$
$A = 0$

$$W_1 = \frac{\frac{1}{M_s}s + \omega_s}{s + \omega_s A} = \frac{0.5s + 5.5}{s}$$

$K(s) = 8$ ✔ $K(s) = 2$ ✘ To meet the nominal performance: $K > 5.5$





Uncertainties and its Cover (MATLAB Program 8)

If for the 4th order plant given in example 6, the uncertainty and performance transfer functions are determined as follows. Plot the both weighting functions in one figure.

$$W_m = \frac{3\,s^2}{s^2 + 2 \times 0.2 \times 45s + 45^2}$$

$$W_S = \frac{15}{s + 0.015}$$

MATLAB Program 8

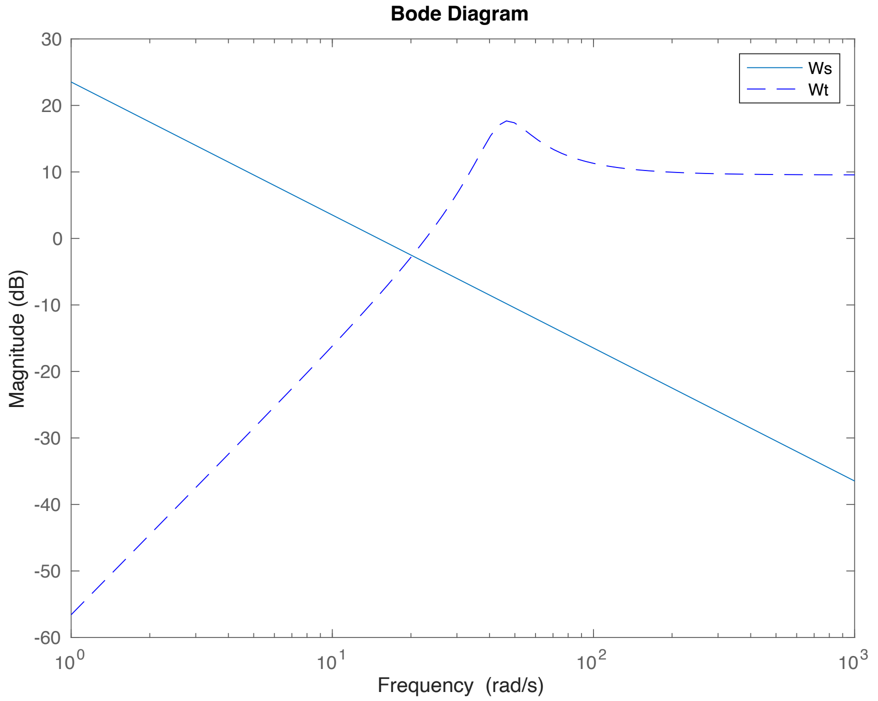
```
%% Plot Uncertainty and Performance Weighting functions

%% Definition of uncertainty weight function Wt (from Program 7)
s = tf('s');
Wm = 3*s^2/(s^2+18*s+45^2);
Wt = Wm; % Wt

%% Determine of performance weight function Ws
s = tf('s');
Ws = 15/(s + 1.5e-2); % Ws

figure(1);
w = logspace(0,3,100); % Definition of frequency vector
bodemag(Ws,Wt,'--',w);
legend('Ws','Wt');
```


Result



Zeros and Poles

When the unstable pole of the plant $P(s)$ is p and its unstable zero is z, $T+S=1$ must be still satisfied. So:

$$S(p)=0, \quad T(p)=1$$
$$S(z)=1, \quad T(z)=0$$

$$S=\frac{1}{1+PK}, \quad T=\frac{PK}{1+PK}$$

$$S(s)=\frac{1}{1+P(s)K(s)}$$

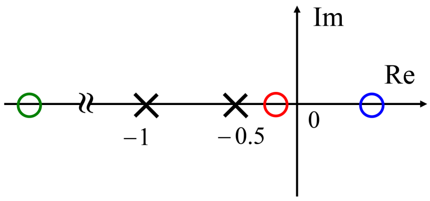
$S(p)=0 \Rightarrow P(p)=\infty$: $\quad S(p)=\frac{1}{1+P(p)K(p)}=0$

$S(z)=1 \Rightarrow P(z)=0$: $\quad S(z)=\frac{1}{1+P(z)K(z)}=\frac{1}{1}=1$

Impact of Zeros

Example:

$$G(s)=\frac{as+1}{(s+1)(2s+1)}$$



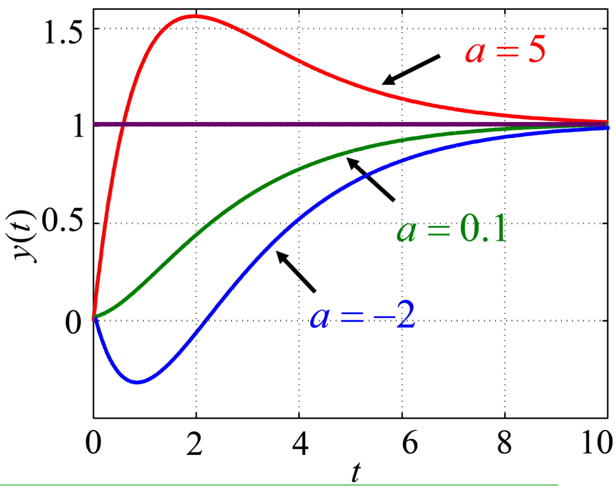
Poles: $-1, \ -0.5$

Zero: $-\frac{1}{a}$

a : Small ⇨ no effect

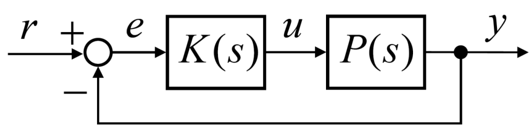
a : Big ⇨ overshoot

$a<0$: ⇨ non-minimum phase



Robust Performance

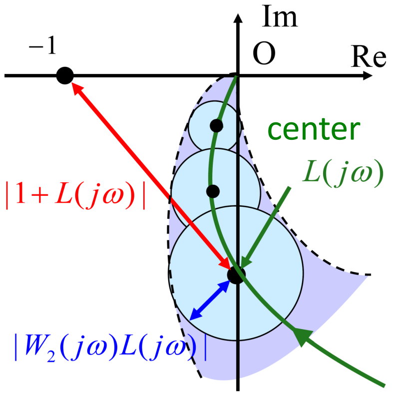
- Uncertain Sensitivity Function:

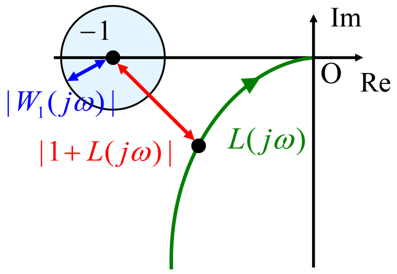


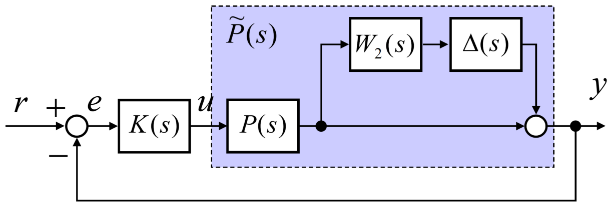
$$\tilde{S} = \frac{1}{1+\tilde{P}K}, \quad \tilde{P} = (1+\Delta W_2)P$$

$\Delta = 0$ ⇨ $\tilde{P} = P, \quad \tilde{S} = S$ (Nominal performance)

Robust Performance = Robust Stability + $|W_1\tilde{S}| < 1, \ ^{\forall}\omega, \ ^{\forall}\tilde{P} \in \mathcal{P}$



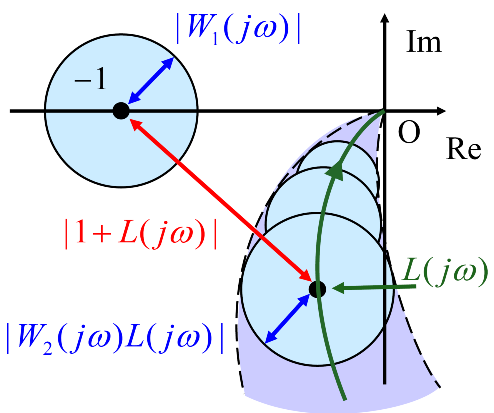




Robust Performance

$$|W_1| + |W_2L| < |1+L| \ ⇨ \ \left|\frac{W_1}{1+L}\right| + \left|\frac{W_2L}{1+L}\right| < 1$$

$$\therefore \ |W_1S| + |W_2T| < 1, \ ^{\forall}\omega$$



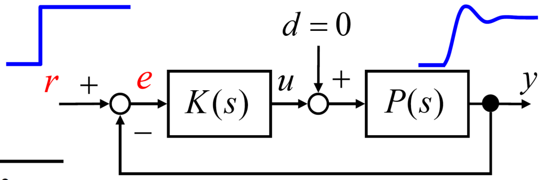
Nominal Stability:	$\phi = D_PD_K + N_PN_K = 0$ (S, T, KS, PS Stable)
Nominal Performance:	$\|W_1S\| < 1, \quad ^{\forall}\omega$
Robust Stability:	$\|W_2T\| < 1, \quad ^{\forall}\omega$
Robust Performance:	$\|W_1S\| + \|W_2T\| < 1, \quad ^{\forall}\omega$
Interpolation Condition:	$S + T = 1, \quad ^{\forall}\omega$

Performance Evaluation

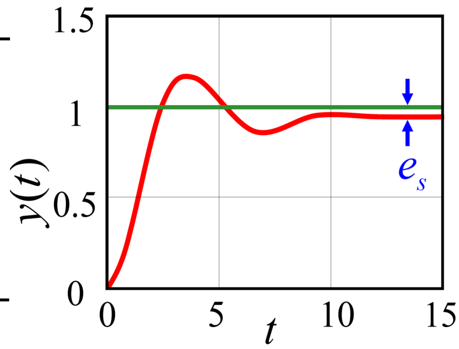
- **Steady-state characteristics**
- Transient characteristics

$$L(s) = P(s)K(s)$$

$$e_s = \frac{1}{1+L(0)} \qquad K_p = L(0)$$

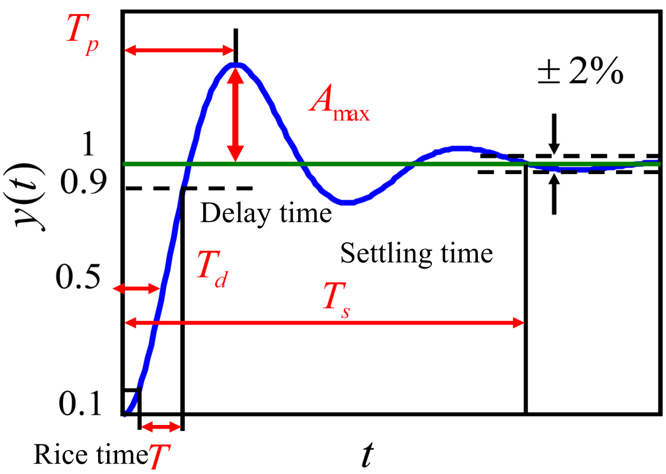


System type	$r(t)=1$	$r(t)=t$	$r(t)=t^2/2$
0 Type	$\frac{1}{1+K_p}$	∞	∞
1 Type	0	$\frac{1}{K_v}$	∞
2 Type	0	0	$\frac{1}{K_a}$



Performance Indices

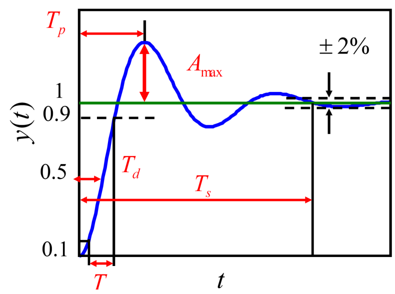
- Steady-state characteristics
- **Transient characteristics**
  - **Time response**
  - Frequency response



Performance Indices

1-order: $T_r \cong 2.2T \quad T_s \cong 4T$

2-order: $T_r \cong \dfrac{1.8}{\omega_n} \quad T_s \cong \dfrac{4}{\zeta\omega_n}$



Example: $T_r < 0.1\text{s}$

1-order: $T_r \cong 2.2T < 0.1\text{s} \quad \dfrac{1}{T} > 22\text{rad/s}$

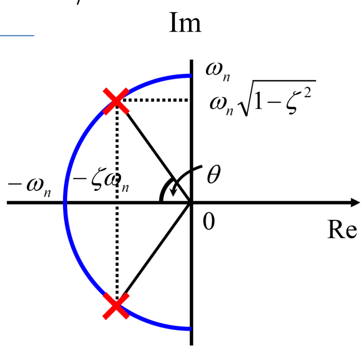
2-order: $T_r \cong \dfrac{1.8}{\omega_n} < 0.1\text{s} \quad \omega_n > 18\text{rad/s}$

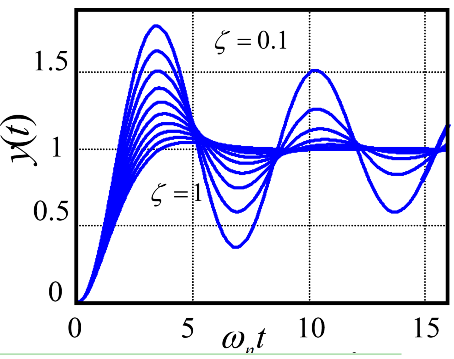
Standard 2-order system:

$$P = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Damping coefficient

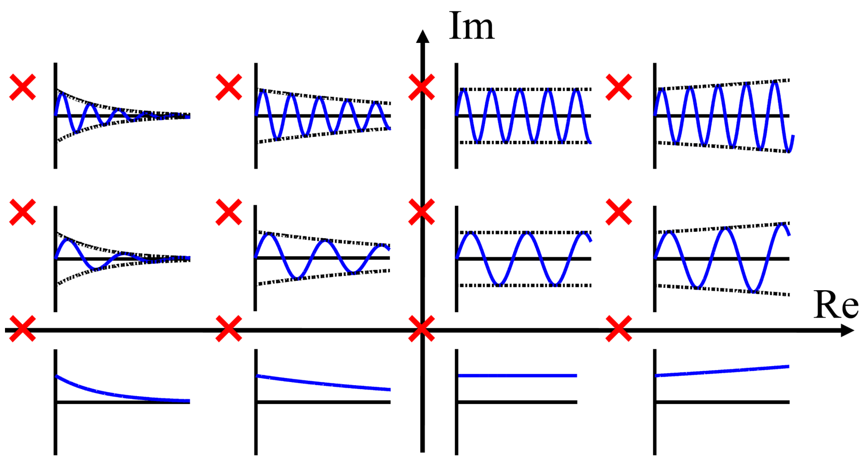
Natural frequency





Performance Indices

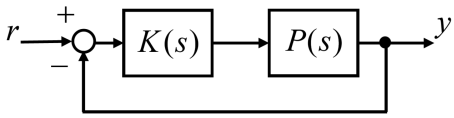
- Steady-state characteristics
- **Transient characteristics**
  - **Time response**
  - Frequency response



Performance Indices

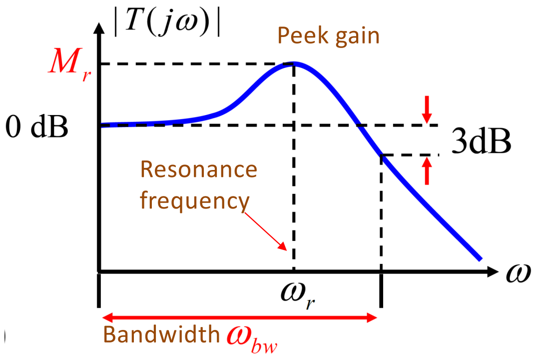
- Steady-state characteristics
- **Transient characteristics**
  - Time response
  - **Frequency response**
    - **Close-loop**
    - Open-loop

Using transfer function



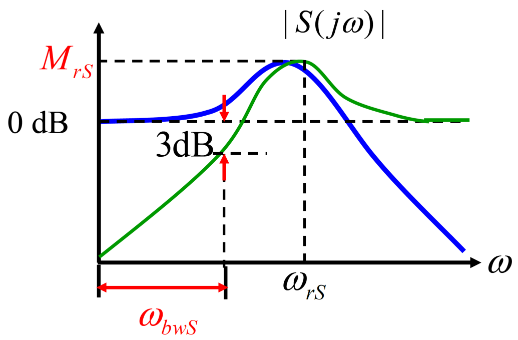
○ **Complementary sensitivity function gain characteristics (T)**

$\omega_{bw}: -3\text{dB} \qquad M_r: M_r = 1.1 \sim 1.5\ (M_r = 1.3)$



○ **Sensitivity function gain characteristics (S)**

$\omega_{bwS}: -3\text{dB} \qquad M_{rS}: M_{rS} < 2$



Performance Indices

- Steady-state characteristics
- **Transient characteristics**
  - Time response
  - **Frequency response**
    - Close-loop
    - **Open-loop**

Using transfer function

Stability margins

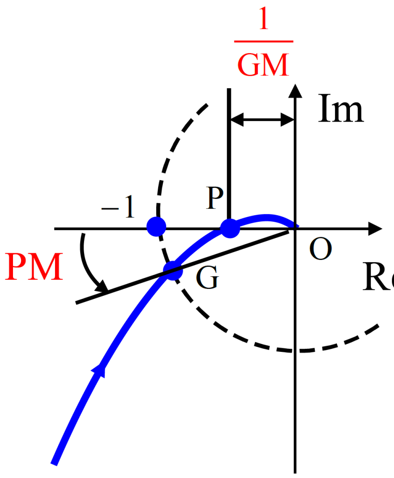
Fast response: $\omega_{gc} \le \omega_{bw}$

$(\text{PM} \le 90°)$

Attenuation characteristic: $\text{PM} \ge 2\sin^{-1}\left(\frac{1}{2M_r}\right)$

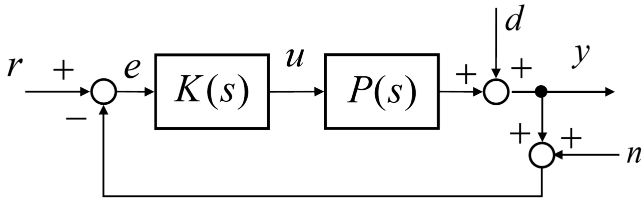
Empirical guidelines

Tracking control:	$\text{PM} = 40 \sim 60°$, $\text{GM} = 10\text{dB} \sim 20\text{dB}$
Fixed value control:	$\text{PM} \ge 20°$, $\text{GM} = 3\text{dB} \sim 10\text{dB}$
2-order system:	$\text{PM} \approx 100 \times \zeta$



Controller Design

Loop Shaping:



- **Nominal performance:** Reduce the sensitivity function S(s)

  - Low sensitivity characteristics (parameter fluctuation) $\Delta_T = S\Delta_P$
  - Disturbance suppression $y = Sd$
  - Reference tracking $e = Sr$ $\quad |W_1 S| < 1$

$$S(s) = \frac{1}{1 + P(s)K(s)}$$

- **Robust performance:** Reduce complementary sensitivity function T(s)

  - Robust stability $|W_2 T| < 1$
  - Noise rejection $y = -Tn$

$$T(s) = \frac{P(s)K(s)}{1 + P(s)K(s)}$$

Controller Design

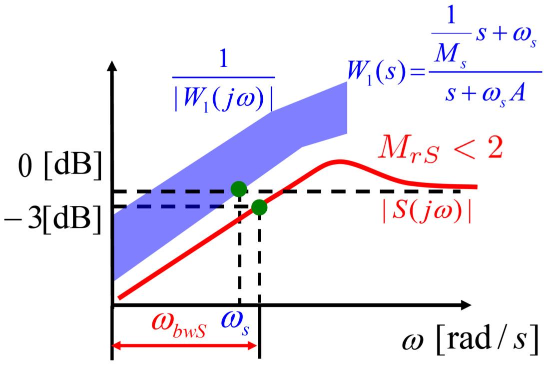
However, due to the following constrain, both functions cannot be close to zero, simultaneously.

$$S(s) + T(s) = 1$$

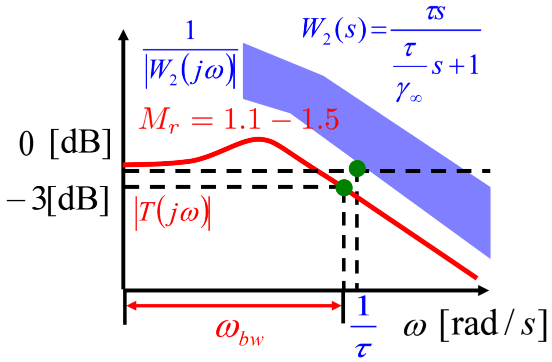
Divide frequency band

Low frequency band: Make S small $\omega \le \omega_s$ using the feedback effect

High frequency band: Make T small $\omega \ge \frac{1}{\tau}$ to cover 100% uncertainty



S and nominal performance



T and Robust stability

Controller Design

Considering $S = \frac{1}{1+L}$, for a small S: $|L| >> 1$

Considering $T = \frac{L}{1+L}$, for a small T: $|L| << 1$

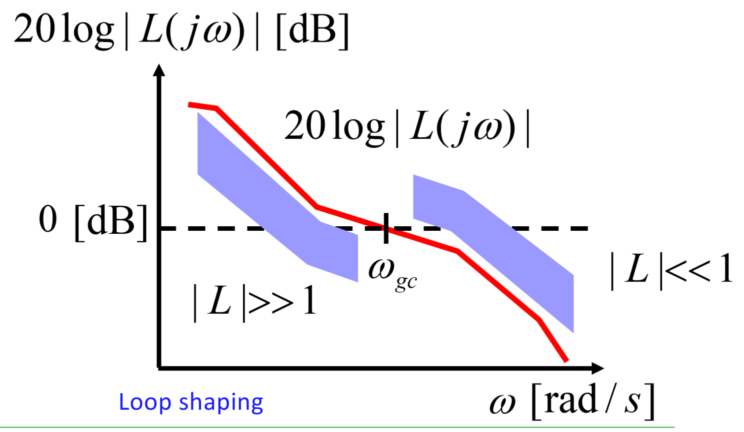
(From closed-loop transfer function to open-loop transfer function)

- Reference tracking control:

  $\text{PM} = 40 \sim 60°, \ \text{GM} = 10\text{dB} \sim 20\text{dB}$

- Fixed value control:

  $\text{PM} \geq 20°, \ \text{GM} = 3\text{dB} \sim 10\text{dB}$



Loop shaping

Controller Design

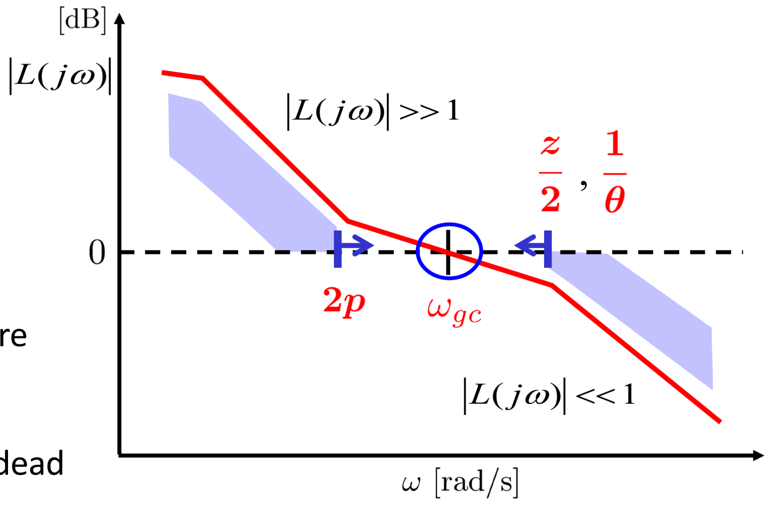
Unstable pole p: $\omega_{gc} > 2p$

Unstable zero z: $\omega_{gc} < \frac{z}{2}$

Dead time θ : $\omega_{gc} < \frac{1}{\theta}$

The unstable zero and the unstable pole are far enough $z/p < 1/6$ or $6 < z/p$

The product of the unstable pole and the dead time is small enough $p\theta < 0.3$



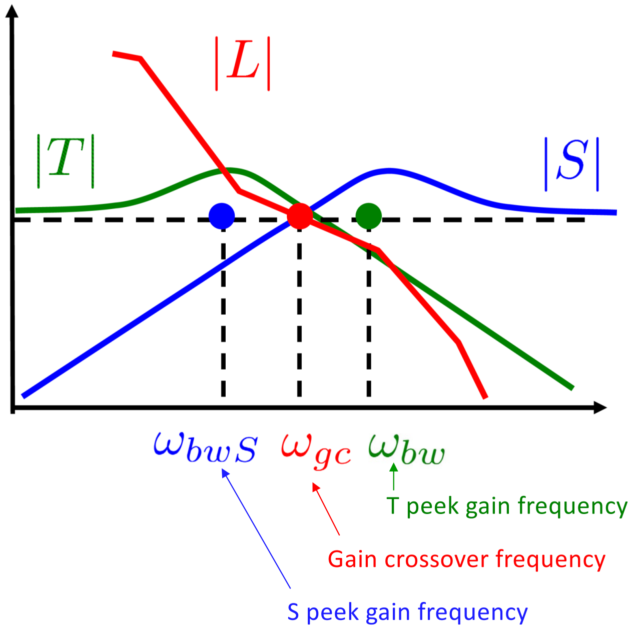
Continue

$$\omega_{bwS} < \omega_{gc} < \omega_{bw}$$
$$(PM < 90^\circ)$$

- S peek gain: $M_{rS} < 2$
- T peek gain: $M_r = 1.1 \sim 1.5 \ \ (M_r = 1.3)$

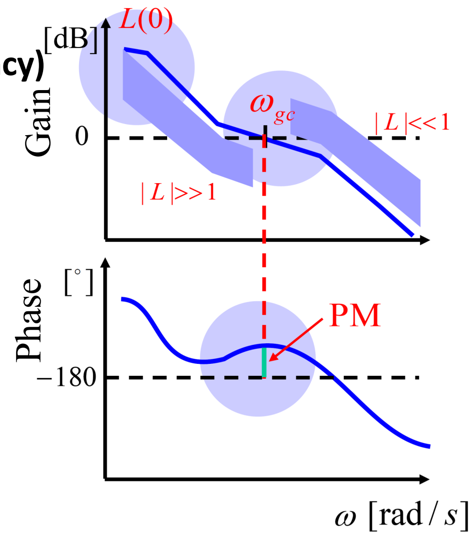
$$GM \geq \frac{M_{rS}}{M_{rS}-1}, \quad PM \geq 2\sin^{-1}\left(\frac{1}{2M_{rS}}\right) \text{ [rad]}$$

$$GM \geq 1 + \frac{1}{M_r}, \quad PM \geq 2\sin^{-1}\left(\frac{1}{2M_r}\right) \text{ [rad]}$$

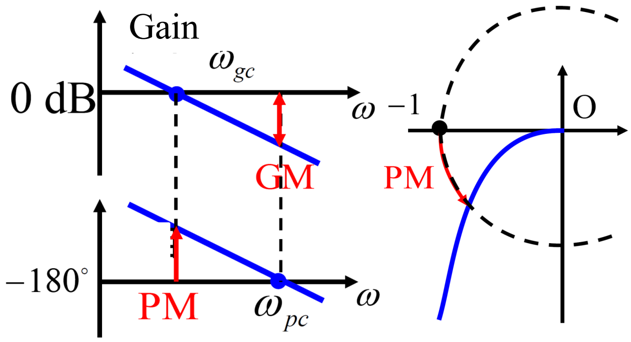


Key Points: Review

- **Steady state characteristics:** Larger gain of L(0) (at low frequency)
- **Fast response:** Increase gain crossover frequency
- **Attenuation characteristic:** Secure phase margin (PM)



[Review] Phase margin

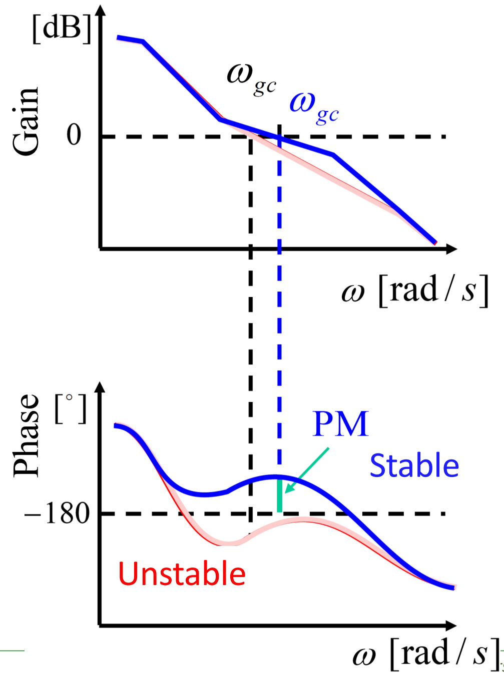


Tracking control: $PM = 40^\circ \sim 60^\circ$

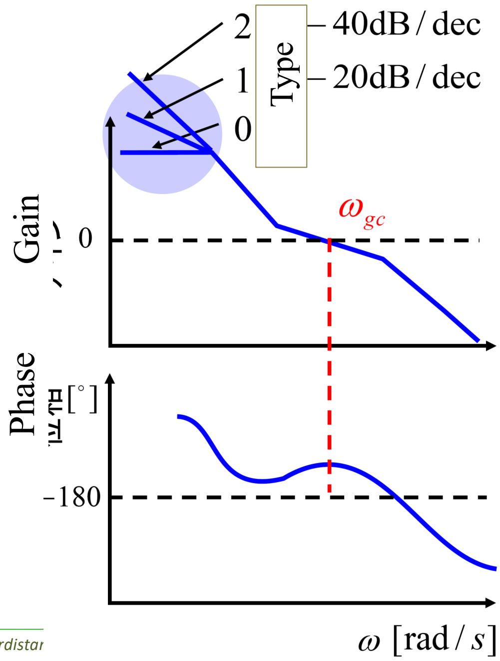
Fixed value control: $PM \geq 20^\circ$

Key Points

- ❑ **Bode diagram and system stability**

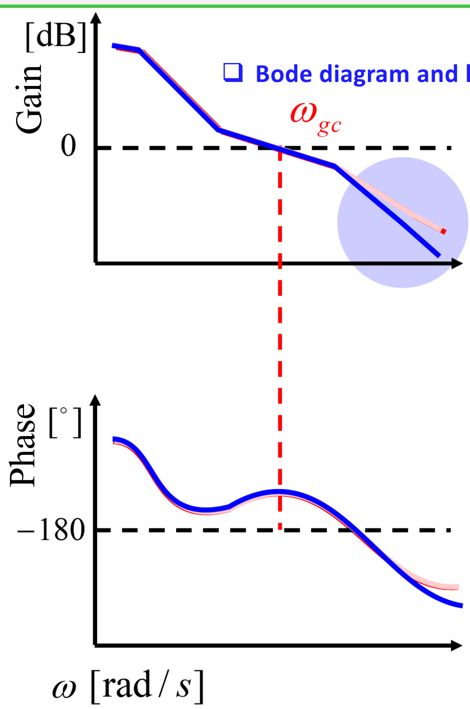


- ❑ **Bode diagram and low frequency behavior** (system type)



Key Points

- ❑ **Bode diagram and high frequency behavior** (Noise reduction)



Project: Report 5

Consider your dynamic system :

1) **Find T(s) and S(s);**
2) **Discuss on the closed-loop performance characteristics;**
3) **Find a performance weighing function;**
4) **Analyze the nominal and robust performance.**

Deadline: The day before next Meeting

Please only use this email address: bevranih18@gmail.com

Thank You!