Robust Control Systems

Structured and Unstructured Uncertainties

Hassan Bevrani

Professor, University of Kurdistan

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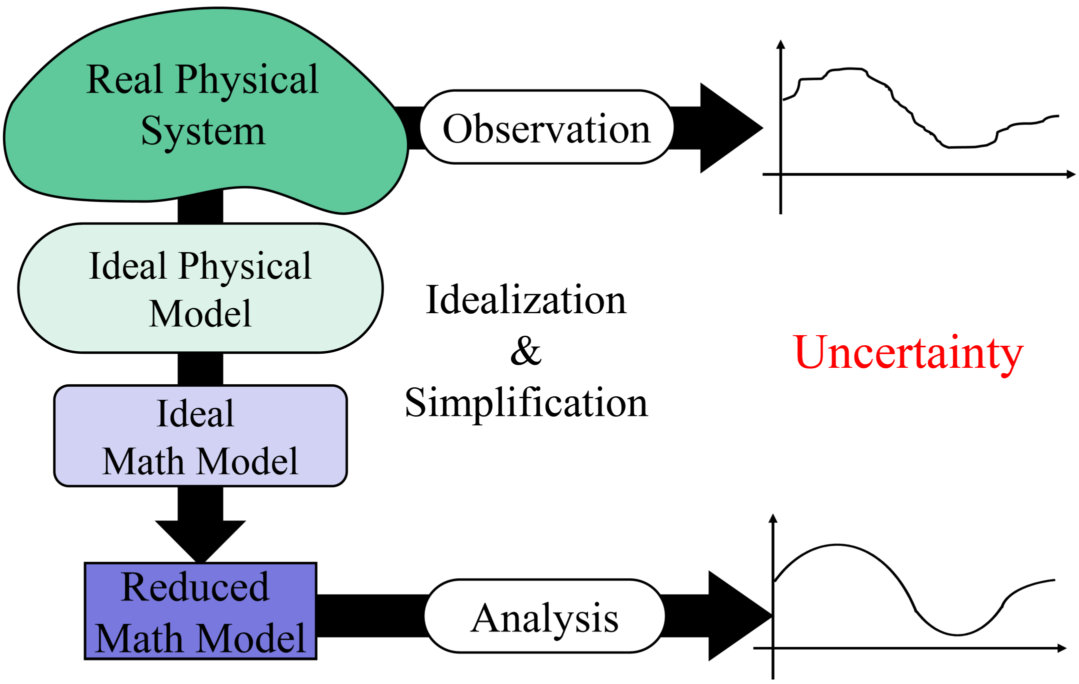
Reference

1. S. Skogestadand I. Postlethwaite, **Multivariable Feedback Control; Analysis and Design**, Second Edition, Wiley, 2005.

2. M. Fujita, **Lecture Notes on Feedback Control Systems**, Tokyo Institute of Technology, 2019.

3. M. Hirata, **Practical Robust Control**, CORONA Press , 2017 (In Japanese).

System and Model



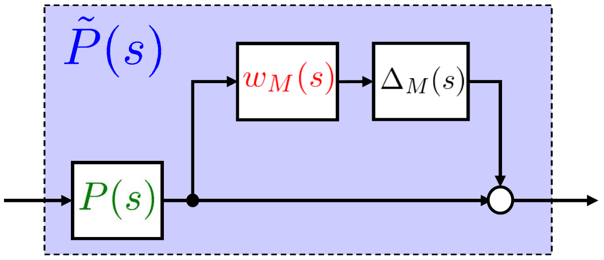
Multiplicative Uncertainty (SISO Systems)

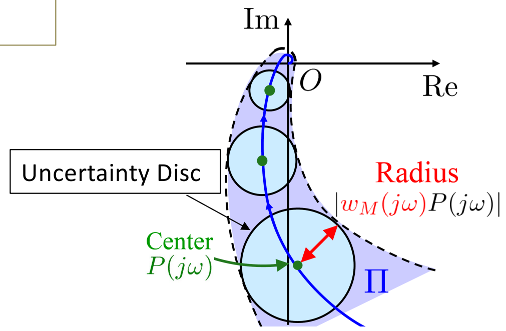
$$\tilde{P}(s) = (1 + \Delta_M(s) w_M(s)) P(s) \qquad \|\Delta_M\|_\infty \le 1$$

Perturbed Plant Model — Uncertainty Weight — Nominal Plant Model

A Set of Plant Models

$$\Pi = \{\tilde{P}(s) | \; \tilde{P}(s) = (1 + \Delta_M(s) w_M(s)) P(s), \; \|\Delta_M\|_\infty \le 1\}$$

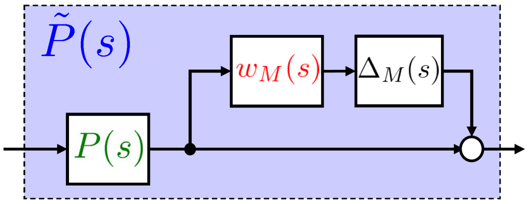




(Ref 1, p. 267)

Obtaining Uncertainty Weight

$$\tilde{P}(s) = (1 + \Delta_M(s) w_M(s)) P(s), \quad \|\Delta_M\|_\infty \le 1$$



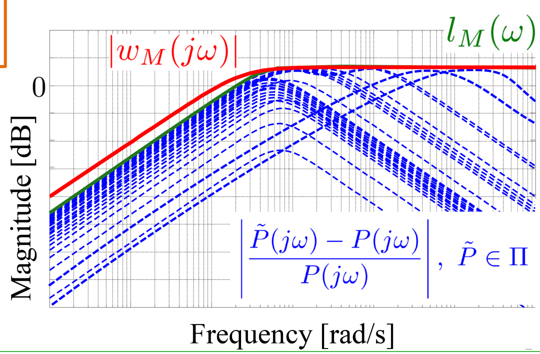
Step 1. Select a nominal model $P(s)$

Step 2. At each frequency, find the smallest radius $l_M(\omega)$ which includes the possible plants $\tilde{P} \in \Pi$:

$$l_M(\omega) = \max_{\tilde{P} \in \Pi} \left| \frac{\tilde{P}(j\omega) - P(j\omega)}{P(j\omega)} \right|$$

Step 3. Choose a (reduced order) weight $w_M(s)$ to cover the set:

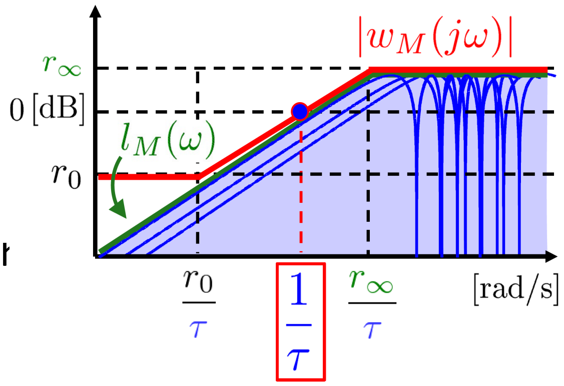
$$|w_M(j\omega)| \ge l_M(\omega), \; \forall \omega$$



(Ref 1, p. 268)

Obtaining Uncertainty Weight

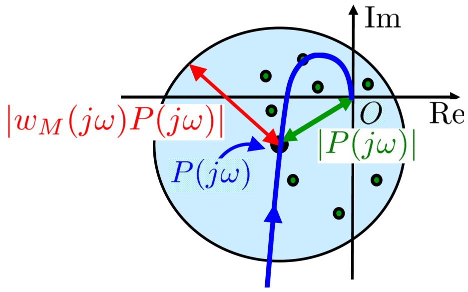
$$w_M(s) = \frac{\tau s + r_0}{\frac{\tau}{r_\infty} s + 1}$$



$1/\tau$: (Approximately) the frequency at which t
relative uncertainty reaches 100%.

r_∞ : Magnitude of w_M at high frequency

r_0 : Relative uncertainty at steady-state



$$|w_M(j\omega)| \geq 1 \;\; (\omega \geq 1/\tau)$$
$$|w_M(j\omega)P(j\omega)| \geq |P(j\omega)|$$

(Ref 1, p. 273)

Example: Time Delay as Uncertainty

$$\tilde{P}(s) = \frac{1}{s+1} e^{-\theta s}, \;\; 0 \leq \theta \leq 1$$

$$\Pi = \{\tilde{P}(s) | \tilde{P}(s) = (1 + \Delta_M(s) w_M(s)) P(s), \; \|\Delta_M\| \leq 1\}$$

Step 1: Nominal Model: $\theta = 0 \Rightarrow P(s) = \frac{1}{s+1}$

Step 2: $l_M(\omega) = \max_{\tilde{P} \in \Pi} \left| \frac{\tilde{P}(j\omega) - P(j\omega)}{P(j\omega)} \right| = \max_{\tilde{P} \in \Pi} |e^{-j\omega\theta} - 1| \leq 2$

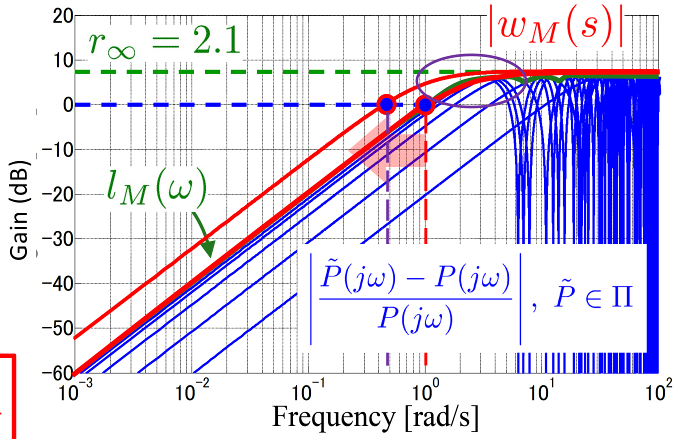
(Ref 1, p. 269)

Continue

Step 2: $l_M(\omega) = \max_{\tilde{P}\in\Pi}\left|\dfrac{\tilde{P}(j\omega)-P(j\omega)}{P(j\omega)}\right| = \max_{\tilde{P}\in\Pi}|e^{-j\omega\theta}-1| \leq 2$

$l_M(\omega) \leq 2$ ➡ $r_\infty = 2.1$

Step 3: $w_M(s) = \dfrac{\tau s}{\frac{\tau}{r_\infty}s+1}$



$\frac{1}{\tau} = 1$ **?** $(\tau = 1)$ $\left(\omega_c \leq \frac{1}{\theta} = 1\right)$

➡ $|w_M(j\omega)| \not\geq l_M(\omega),\ \forall\omega$

$\frac{1}{\tau} = 0.48$ $(\tau = 2.1)$

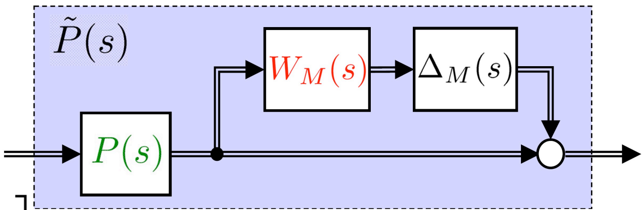
$w_M(s) = \dfrac{2.1s}{s+1}$

(Ref 1, p. 269)

Representing Uncertainty (MIMO Systems)

Multiplicative (Output) Uncertainty

$\Pi_0 = \{\tilde{P}(s)|\ \tilde{P}(s) = (I+\Delta_M(s)W_M(s))P(s),\ \ \|\Delta_M\|_\infty \leq 1\}$



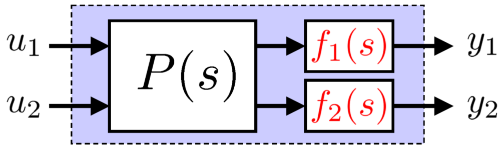
Uncertainty Weight $W_M(s)$

$$\begin{bmatrix} w_{M11}(s) & \cdots & w_{M1n}(s) \\ \vdots & \ddots & \vdots \\ w_{Mn1}(s) & \cdots & w_{Mnn}(s) \end{bmatrix}, \begin{bmatrix} w_{M1}(s) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & w_{Mn}(s) \end{bmatrix}$$

$$W_M(s) = w_M(s)I = \begin{bmatrix} w_M(s) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & w_M(s) \end{bmatrix},\ w_M(s) = \frac{\tau s + r_0}{\frac{\tau}{r_\infty}s+1}$$

Example: Spinning Satellite

Uncertain Plant Model $P(s) = \begin{bmatrix} \frac{s-100}{s^2+100} & \frac{10s+10}{s^2+100} \\ \frac{-10s-10}{s^2+100} & \frac{s-100}{s^2+100} \end{bmatrix}$

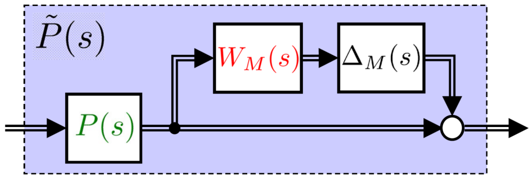


$$\tilde{P}(s) = \begin{bmatrix} f_1(s) & 0 \\ 0 & f_2(s) \end{bmatrix} P(s)$$

$$f_i(s) = k_i \frac{-\frac{\theta_i}{2}s + 1}{\frac{\theta_i}{2}s + 1}, \quad i = 1, 2$$

Gain Margin: $0.8 \le k_i \le 1.2$ $(\pm 20\%, \text{GM} = 2\text{dB})$

Delay Margin: $0 \le \theta_i \le 0.02$



Multiplicative (Output) Uncertainty

$$\Pi_0 = \{\tilde{P}(s) |\ \tilde{P}(s) = (I + \Delta_M(s) W_M(s)) P(s),\ \|\Delta_M\|_\infty \le 1\}$$

(Ref 1, p. 295)

Continue

Step 1) Nominal Model: $k_i = 1,\ \theta_i = 0\ ,\ i = 1, 2$ ⇨ $P(s) = P(s)$

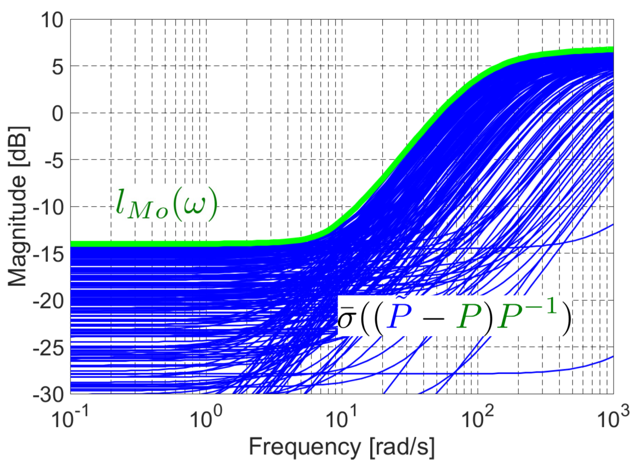
Step 2) $l_{Mo}(\omega) = \max_{\tilde{P} \in \Pi_0} \bar{\sigma}((\tilde{P}(j\omega) - P(j\omega)) P^{-1}(j\omega))$

MATLAB Command

```
k1 = ureal('k1',1,'Per',[-20 20]);
k2 = ureal('k2',1,'Per',[-20 20]);
L1 = ureal('L1',0.01,'Range',[0 0.02]);
L2 = ureal('L2',0.01,'Range',[0 0.02]);
f1 = k1*tf([-L1/2 1],[L1/2 1]);
f2 = k2*tf([-L2/2 1],[L2/2 1]);
f = [f1 0;0 f2];
farray = usample(f,100);
```

100 randomly generated parameters

```
Parray=farray*Pnom;
Pfarray=frd(Parray,logspace(-1,3,100));
Eo=(Pfarray-Pnom)*inv(Pnom);
figure
sigma(Eo,'b-');
hold on; grid on;
```



(Ref 1, p. 295)

Continue

Step 3) $W_M(s) = w_M(s)I_2$, $w_M(s) = \dfrac{\tau s + r_0}{\frac{\tau}{r_\infty}s + 1}$ $\quad |w_M(j\omega)| \geq l_{Mo}(\omega), \forall\omega$

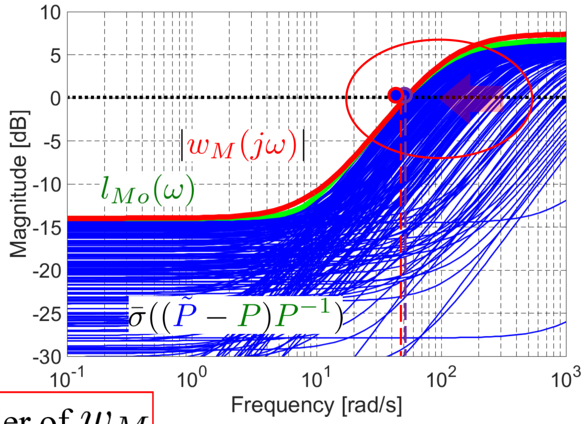
$r_0 = 0.2,\ r_\infty = 2.3$

$\dfrac{1}{\tau} = 50 \quad (\tau = 0.02)$ **?**

$\left(\omega_c \leq \dfrac{1}{\theta} = 50\right)$

$\Rightarrow \dfrac{1}{\tau} = 48 \quad (\tau = 0.021)$

$\Rightarrow w_M(s) = \dfrac{0.021s + 0.2}{0.0091s + 1}$



Manual Fitting

MATLAB Command

```
r0 = 0.2; rinf = 2.3; tau = 0.021;
wM = tf([tau r0], [tau/rinf 1]);
WM = eye(2)*wM;
sigma(WM,'r');
```

Automatic Fitting

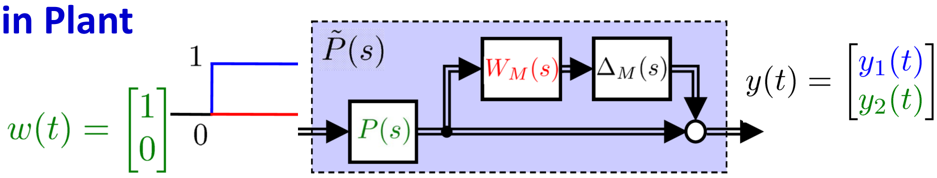
MATLAB Command

Order of w_M

```
[Usys,uInfo] = ucover(Parray,Pnom,1,'OutputMult');
sigma(uInfo.W1opt,'g-');
wM = uInfo.W1; WM = eye(2)*wM;
sigma(WM,'r');
```

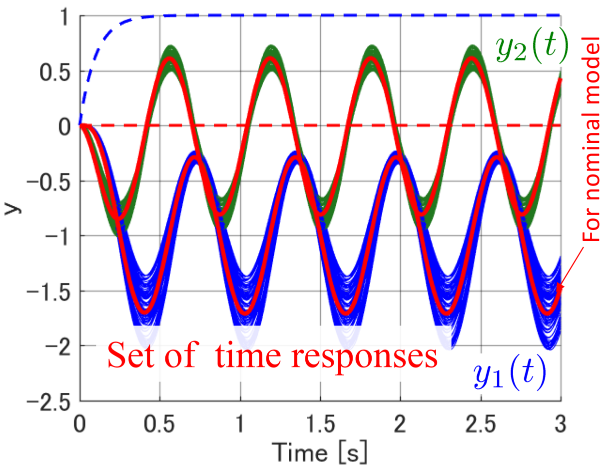

More Continue

Time Responses for Uncertain Plant

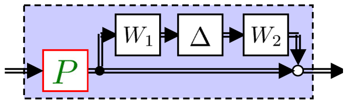
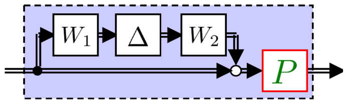
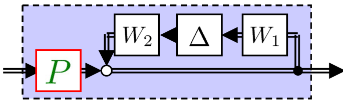
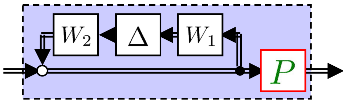
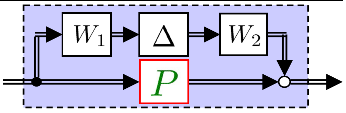
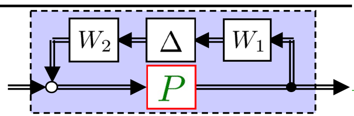


MATLAB Command

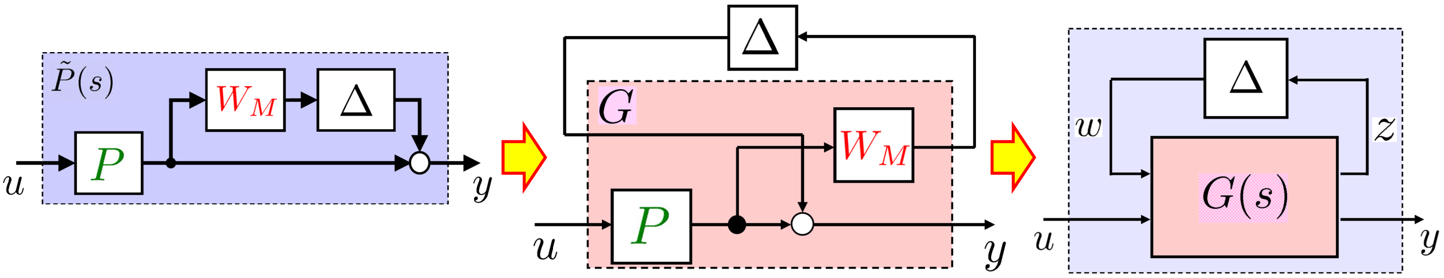
```
time = 0:0.01:3;
step_ref = ones(1,length(time));
Filter = tf(1,[0.1 1]);
step_ref_filt = lsim(Filter,step_ref,time);
ref = [step_ref_filt'; zeros(1,length(time))];

figure
hold on; grid on;
Parray=farray*Pnom;
for i = 1 : 100
    [yhi,t] = lsim(Parray(:,:,i),ref,time);
    plot(t,yhi(:,1),'b-');
    plot(t,yhi(:,2),'g-');
end

[yhi1,t] = lsim(Pnom,ref,time);
plot(t,yhi1,'r-');
plot(time,ref,'g-.');
```



Unstructured Uncertainty Modeling

Unstructured Uncertainty	Representation	
Multiplicative (Output)		$(I + W_2 \Delta W_1)P$
Multiplicative (Input)		$P(I + W_2 \Delta W_1)$
Inverse Multip. Output		$(I - W_2 \Delta W_1)^{-1} P$
Inverse Multip. Input		$P(I - W_2 \Delta W_1)^{-1}$
Additive		$P + W_2 \Delta W_1$
Inverse Additive		$P(I - W_2 \Delta W_1 P)^{-1}$

Linear Fractional Transformation (LFT)



$$w = \Delta z \;, \quad \|\Delta\|_\infty \le 1 \qquad \begin{bmatrix} z \\ y \end{bmatrix} = G(s) \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

Upper LFT (ULFT):

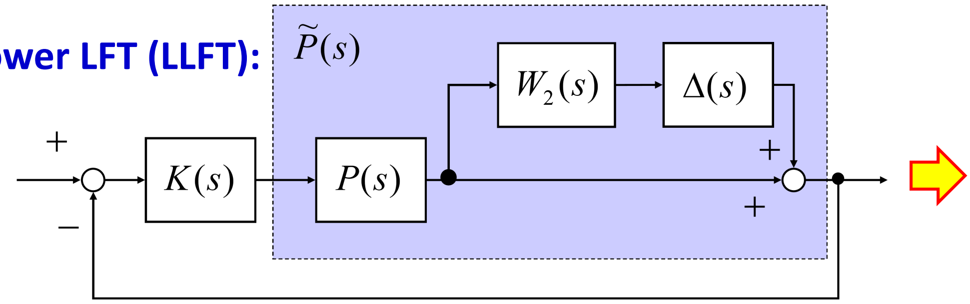
$$y = F_u(G, \Delta) u$$
$$F_u(G, \Delta) = G_{22} + G_{21} \Delta (I - G_{11} \Delta)^{-1} G_{12}$$

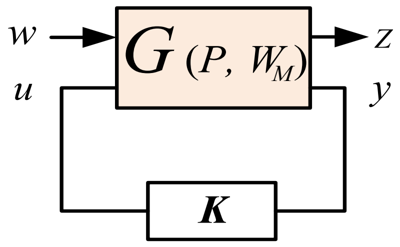
?

(Ref 1, pp. 113, 543)

Linear Fractional Transformation (LFT)

Lower LFT (LLFT):





$$G\ (P,\ W_M) = ?$$

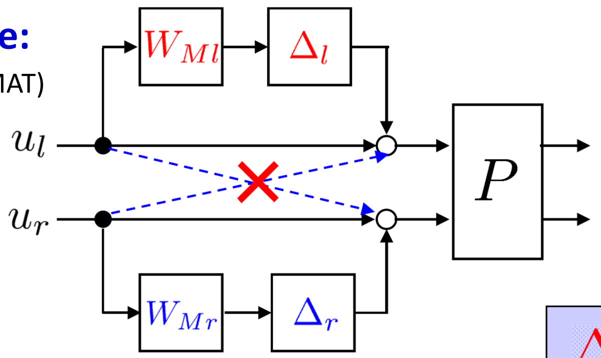
$$W_2 \Delta W_1 \neq W_1 \Delta W_2 \qquad \|\Delta\|_\infty \leq 1$$

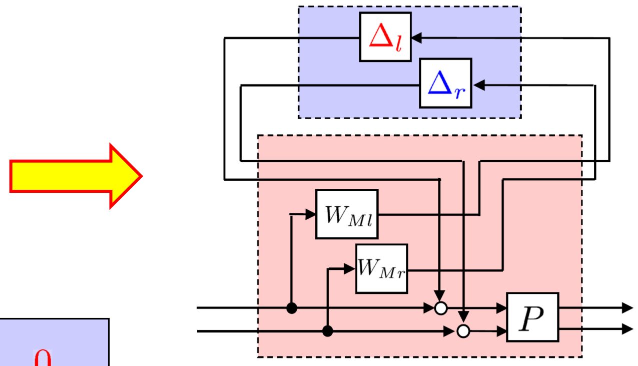
(Ref 1, pp. 113, 543)

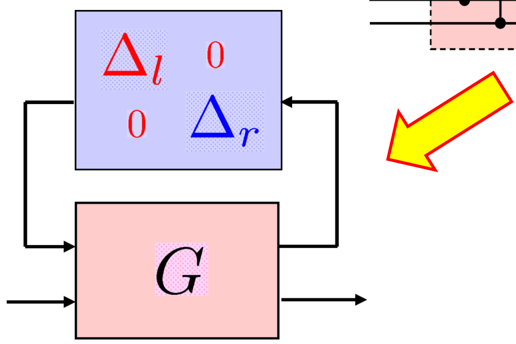
Structured Uncertainty

Example:

(NASA HIMAT)





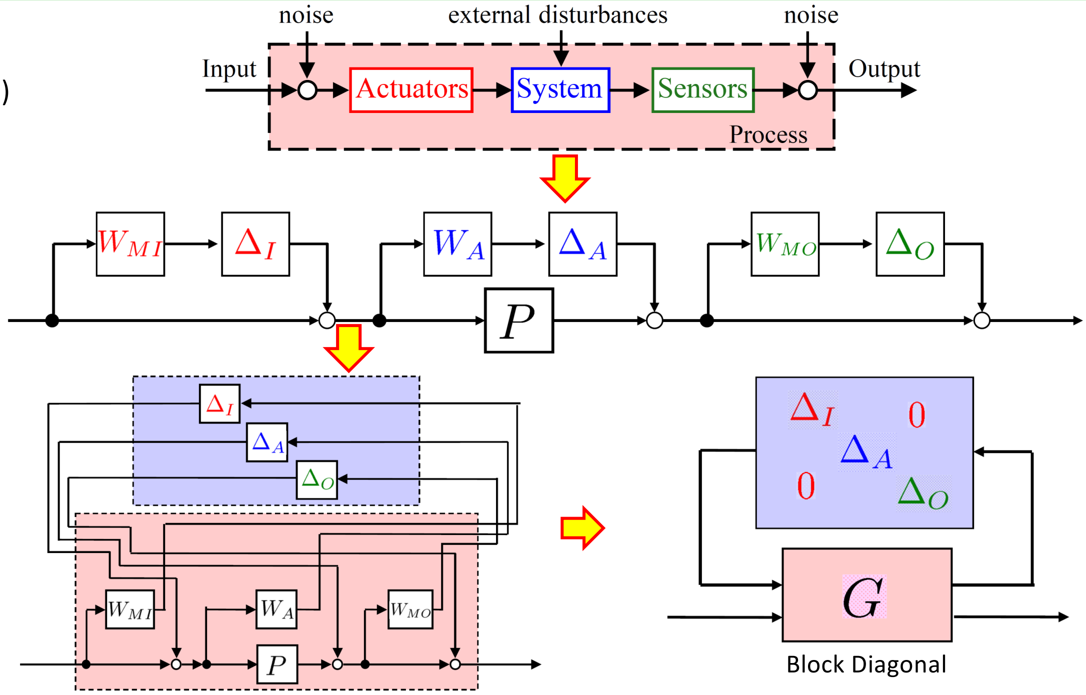


Block Diagonal

Stability Margin in Multivariable Systems, A.E.Bryson. Jr., IEEE TAC, 22 -5, 1977

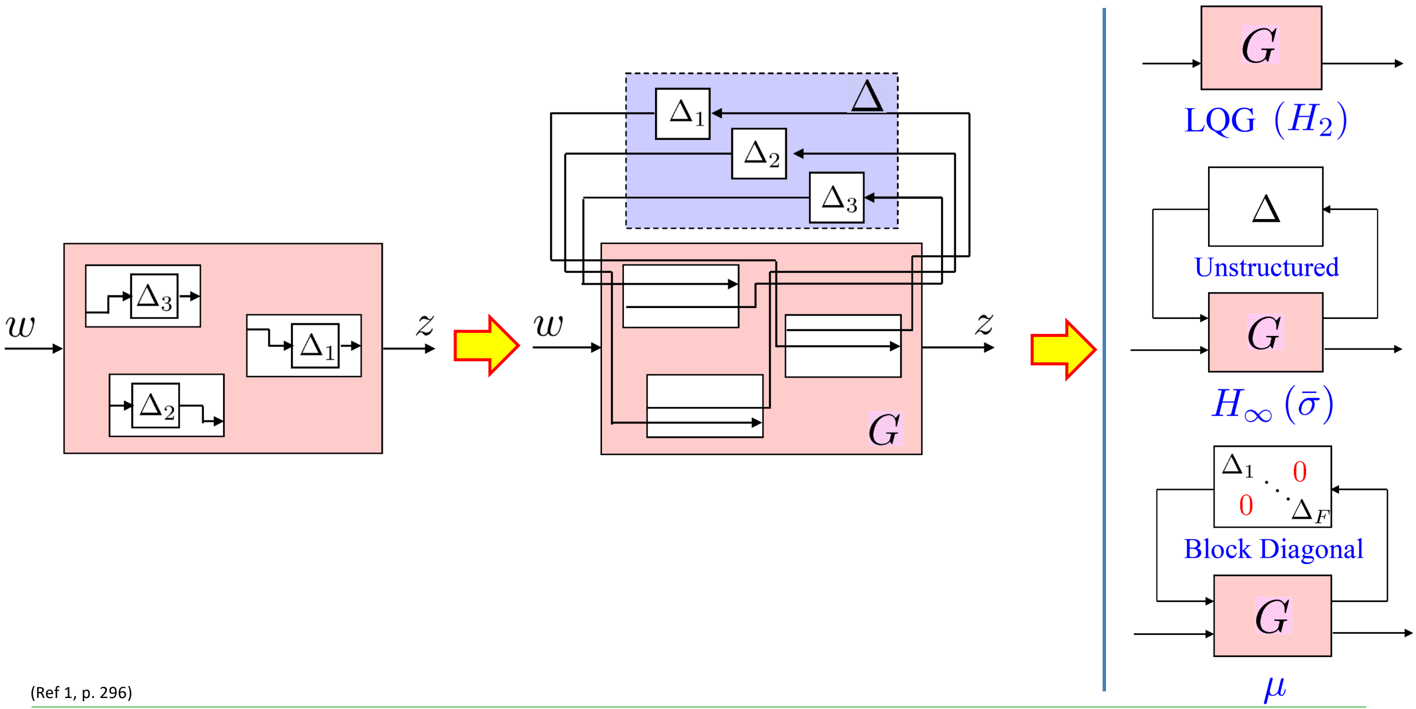
Structured Uncertainty

Example:
(X-29 Aircraft)



(Ref 1, p. 296)

Uncertainty in Various Control Design



(Ref 1, p. 296)

Structured Uncertainty Example (MATLAB Program 9)

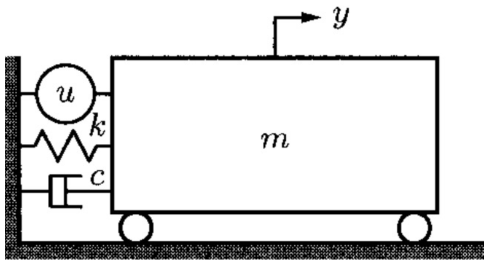
In the following Mass-Spring-Damper system,

a) Find the state space model and integrator block diagram,

b) Considering uncertain parameters m and k, find generalized plant (G), uncertainty block (∆), and LFT,

c) Write a MATLAB program to solve (a) and (b).

$$m\ddot{y} + c\dot{y} + ky = u$$



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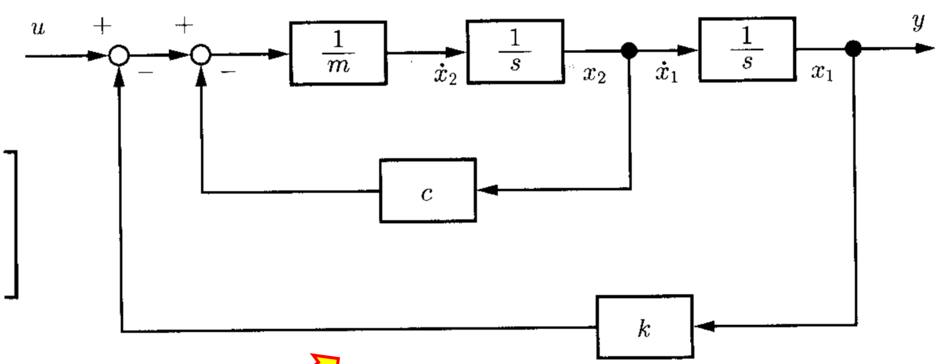
a) State space model and integrator block diagram:

$$m\ddot{y} + c\dot{y} + ky = u$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$$



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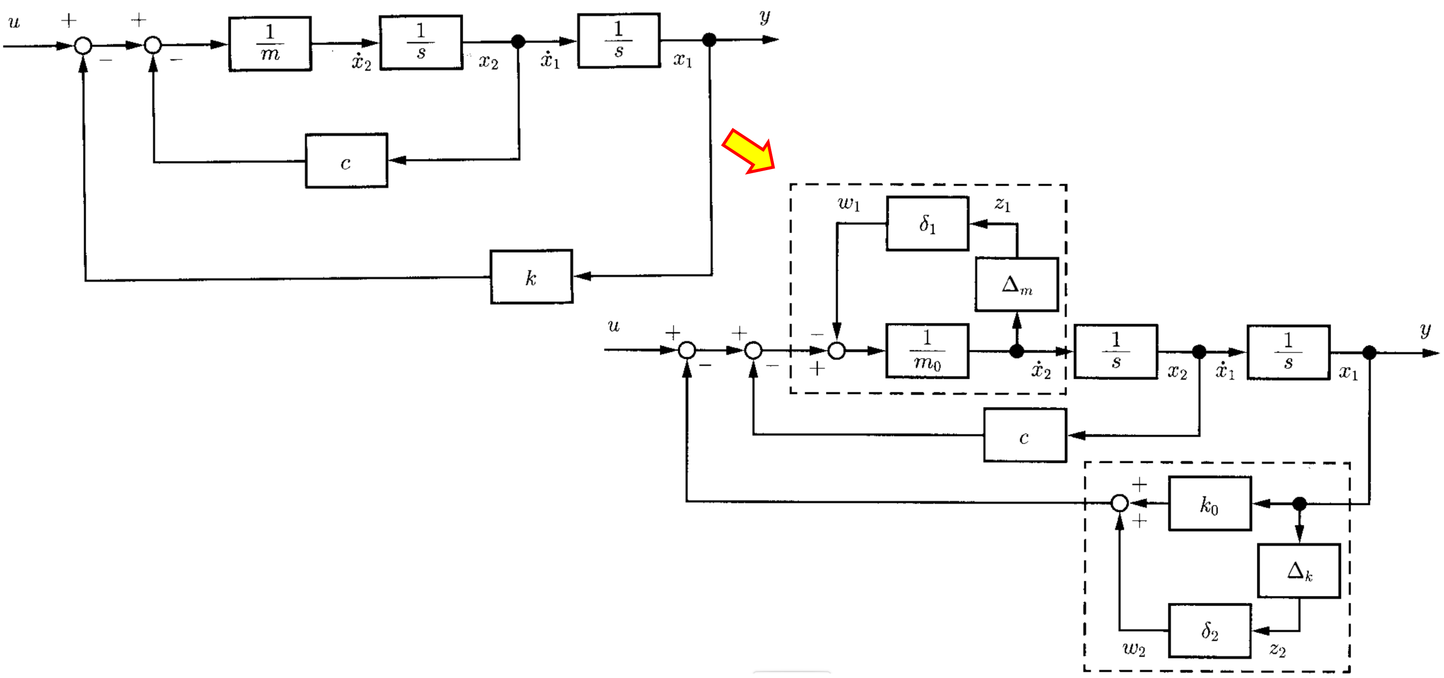
b) Find generalized plant (G), uncertainty block (Δ), and LFT:

$$m_L \leq m \leq m_H, \quad k_L \leq k \leq k_H \quad \Rightarrow \quad \begin{cases} m_0 := \dfrac{m_H + m_L}{2}, & \Delta_m := \dfrac{m_H - m_L}{2} \\ k_0 := \dfrac{k_H + k_L}{2}, & \Delta_k := \dfrac{k_H - k_L}{2} \end{cases}$$

Multiplicative uncertainty: $m = m_0 + \Delta_m \delta_1, \quad k = k_0 + \Delta_k \delta_2$

$$\Rightarrow \quad \frac{1}{m} = \frac{1}{m_0 + \Delta_m \delta_1}$$

Continue



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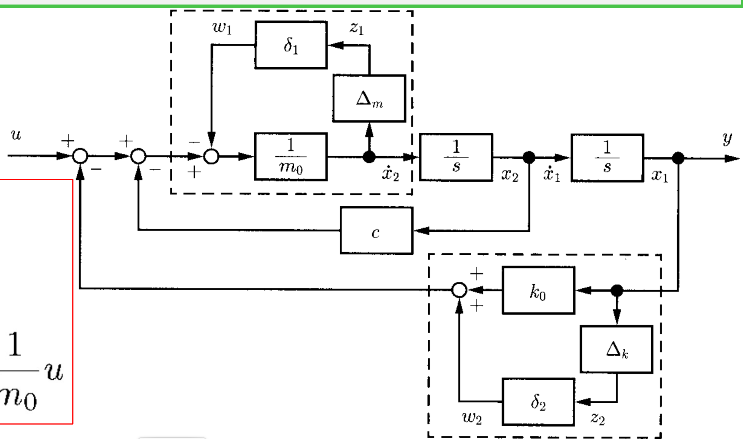


$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{m_0}[u - (k_0\, x_1 + w_2) - c\, x_2 - w_1] \\
&= -\frac{k_0}{m_0}x_1 - \frac{c}{m_0}x_2 - \frac{1}{m_0}w_1 - \frac{1}{m_0}w_2 + \frac{1}{m_0}u
\end{aligned}$$

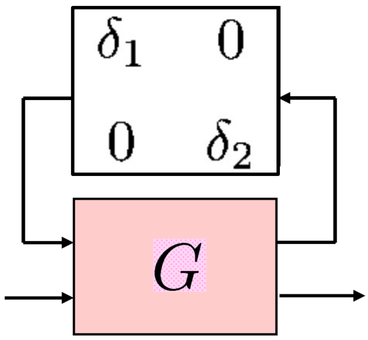
$$\begin{aligned}
z_1 &= \Delta_m \dot{x}_2 \\
&= -\Delta_m\frac{k_0}{m_0}x_1 - \Delta_m\frac{c}{m_0}x_2 - \frac{\Delta_m}{m_0}w_1 - \frac{\Delta_m}{m_0}w_2 + \frac{\Delta_m}{m_0}u \\
z_2 &= \Delta_k x_1 \\
y &= x_1
\end{aligned}$$

Continue

Using input vector $[w_1,\ w_2,\ u]^T$ and output vector $[z_1,\ z_2,\ y]^T$:

$$\begin{bmatrix} z_1 \\ z_2 \\ \hdashline y \end{bmatrix} = G \begin{bmatrix} w_1 \\ w_2 \\ \hdashline u \end{bmatrix}$$

$$G := \left[\begin{array}{cc|cc:c}
0 & 1 & 0 & 0 & 0 \\
-\frac{k_0}{m_0} & -\frac{c}{m_0} & -\frac{1}{m_0} & -\frac{1}{m_0} & \frac{1}{m_0} \\
\hline
-\Delta_m\frac{k_0}{m_0} & -\Delta_m\frac{c}{m_0} & -\frac{\Delta_m}{m_0} & -\frac{\Delta_m}{m_0} & \frac{\Delta_m}{m_0} \\
\Delta_k & 0 & 0 & 0 & 0 \\
\hdashline
1 & 0 & 0 & 0 & 0
\end{array}\right]$$



$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

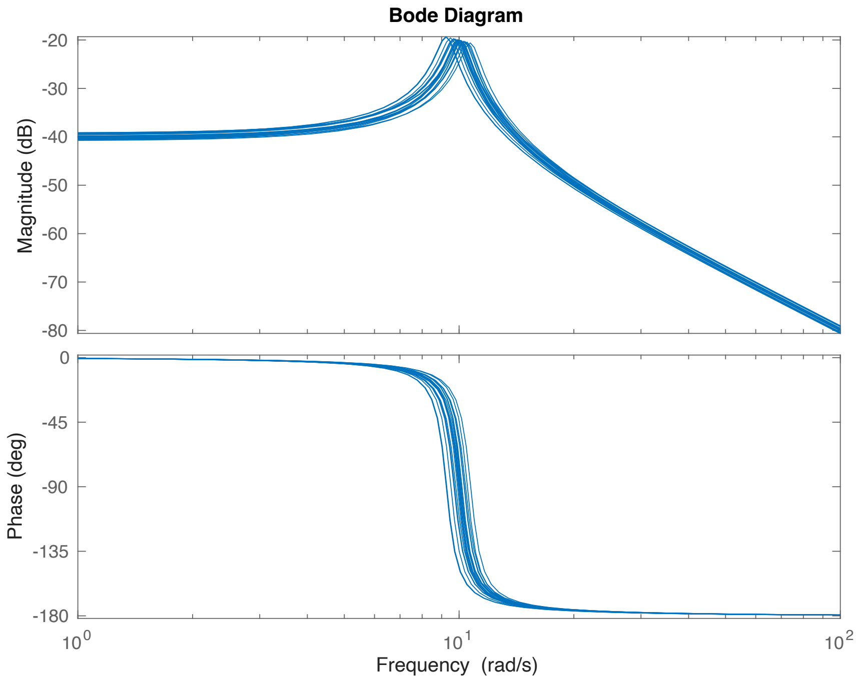
$$y = \mathcal{F}_u(G, \Delta)u, \qquad \Delta = \mathrm{diag}[\delta_1,\ \delta_2]$$

MATLAB Program 9

```
%% Structured Uncertainty Example (MATLAB Program 9)

close all; clear all;
%% Naminal values
m0 = 1; k0 = 100; c = 1;
%% Perturbation range
Delta_m = m0*0.1; Delta_k = k0*0.1;
%% Normalized real perturbation
delta_1 = ureal('delta_1',0); delta_2 = ureal('delta_2',0);
Delta = blkdiag(delta_1,delta_2);
%% G state-space model
A = [ 0, 1 ; -k0/m0, -c/m0 ];
B = [ 0, 0, 0 ; -1/m0, -1/m0, 1/m0 ];
C = [ -k0/m0*Delta_m, -c/m0*Delta_m ; Delta_k, 0 ; 1, 0 ];
D = [ -Delta_m/m0, -Delta_m/m0, Delta_m/m0 ; 0, 0, 0 ; 0, 0, 0 ];
G = ss(A,B,C,D);
%% LFT calculation
P = lft(Delta,G);
%% Bode plot
bode(P)
```


Results



Results

```
>> P
Uncertain continuous-time state-space model with 1 outputs, 1 inputs, 2 states.
The model uncertainty consists of the following blocks:
  delta_1: Uncertain real, nominal = 0, variability = [-1,1], 1 occurrences
  delta_2: Uncertain real, nominal = 0, variability = [-1,1], 1 occurrences
```

```
>> P.NominalValue
ans =
A =
     x1   x2
  x1   0    1
  x2 -100   -1
 B =
    u1
  x1  0
  x2  1
C =
    x1 x2
  y1  1  0
D =
    u1
  y1  0
```


An Alternative Program

```
%% An Alternative for Program 9
close all; clear all
%% Nominal Model
m0 = 1;
k0 = 100;
c = 1;
%% Define Real Uncertainties
m = ureal('m',m0,'percent',10);
k = ureal('k',k0,'percent',10);
%% State-Space Realization
A = [ 0, 1 ;
-k/m, -c/m ];
B = [ 0;
1/m ];
C = [ 1, 0 ];
D = [ 0 ] ;
P = ss(A,B,C,D);
%% LFT
[G,Delta,BlkStruc,NormUNC] = lftdata(P);
size(Delta)
NormUNC{:}
```

lftdata

Decomposes an uncertain matrix or system.

Results

```
>> size(Delta)
```

Uncertain matrix with 3 rows, 3 columns, and 3 blocks. delta_2: Uncertain real, nominal = 0, variability = [-1,1], 1 occurrences

```
>> NormUNC{:}
```

Uncertain real parameter "kNormalized" with nominal value 0 and variability [-1,1].
Uncertain real parameter "mNormalized" with nominal value 0 and variability [-1,1].
Uncertain real parameter "mNormalized" with nominal value 0 and variability [-1,1].

```
>> P
```

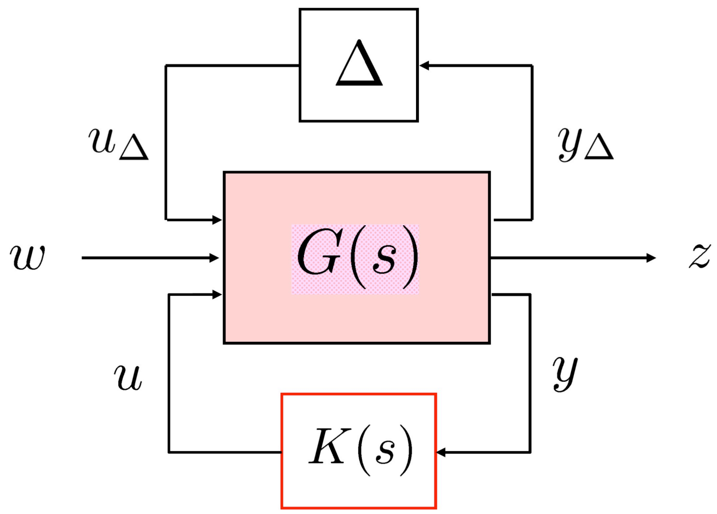
Uncertain continuous-time state-space model with 1 outputs, 1 inputs, 2 states.
The model uncertainty consists of the following blocks:
k: Uncertain real, nominal = 100, variability = [-10,10]%, 1 occurrences
m: Uncertain real, nominal = 1, variability = [-10,10]%, 2 occurrences

Overall Robust Control Framework

$\|\Delta\|_\infty \leq 1$

$G(s)$: Generalized Plant

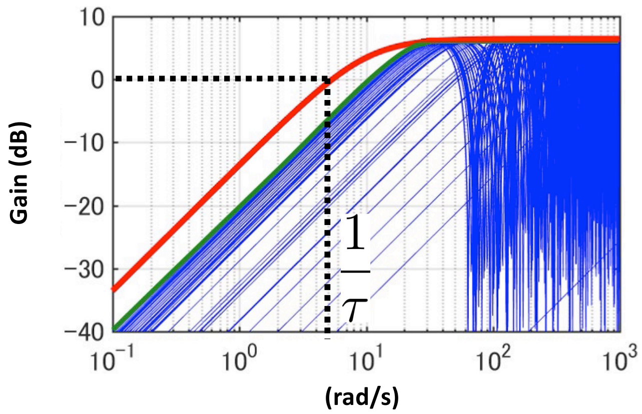
$K(s)$: Controller



(Ref 1, pp. 12, 289)

Uncertainties

1. Unstructured



2. Structured:

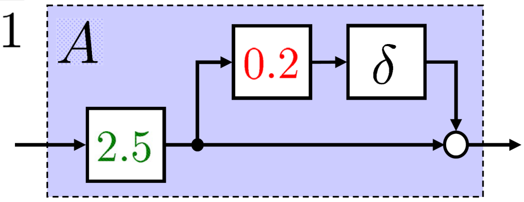
1. Parametric
2. Nonparametric

$2 \le A \le 3$

$\boldsymbol{\varphi}(s)$

Parametric Uncertainty: Individual Modeling

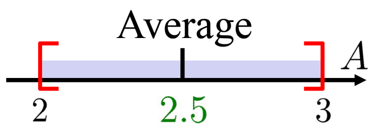
Example: First Order Plant Model $P(s) = \dfrac{A}{Ts+1}$



Case 1: Uncertain Parameter $2 \le A \le 3$

$$A = 2.5(1 + \delta \cdot 0.2), \;\; |\delta| \le 1$$

$$\left[\begin{array}{l} \text{Nominal Value} \;\; 2.5 \\ \text{Uncertainty} \;\; \pm 20\% \;\;\; \text{any} \;\; |\delta| \le 1 \end{array}\right]$$



[Ex.] $A = 3$ for $\delta = 1$

$A = 2$ for $\delta = -1$

Parametric Uncertainty: State-Space Modeling

Example:

$$A_p = \begin{bmatrix} -2-\alpha & \alpha-\beta \\ \alpha+2\beta & -\alpha \end{bmatrix} \qquad \begin{matrix} \alpha = 1 + w_1\delta_1 & |\delta_1| \le 1 \\ \beta = 3 + w_2\delta_2 & |\delta_2| \le 1 \end{matrix}$$

$$\Rightarrow A_p = \underbrace{\begin{bmatrix} -3 & -2 \\ 7 & -1 \end{bmatrix}}_{A} + \delta_1 \underbrace{\begin{bmatrix} -w_1 & w_1 \\ w_1 & -w_1 \end{bmatrix}}_{E_1} + \delta_2 \underbrace{\begin{bmatrix} 0 & -w_2 \\ 2w_2 & 0 \end{bmatrix}}_{E_2}$$

$\text{rank}(E_1) = 1$ $\qquad$ $\text{rank}(E_2) = 2$

$$E_1 + E_2 = \underbrace{\left[\begin{array}{c|cc} -w_1 & 0 & -w_2 \\ w_1 & 2w_2 & 0 \end{array}\right]}_{W_2} \underbrace{\left[\begin{array}{c|cc} \delta_1 & 0 & 0 \\ \hline 0 & \delta_2 & 0 \\ 0 & 0 & \delta_2 \end{array}\right]}_{\Delta} \underbrace{\left[\begin{array}{cc} 1 & -1 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right]}_{W_1}$$

$$\Rightarrow A_p = A + E_1 + E_2 = A + W_2 \Delta W_1$$

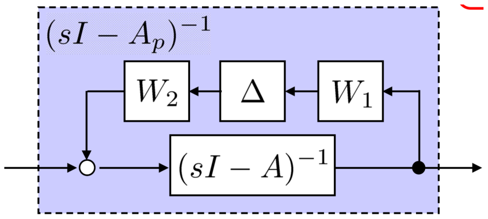
(Ref 1, p. 292)

Parametric Uncertainty: State Space

$$\Phi(s) = (sI - A)^{-1}$$

$$(sI - A_p)^{-1} = (sI - A - W_2\Delta W_1)^{-1}$$

$$= (I - \Phi(s) W_2 \Delta W_1)^{-1} \Phi(s)$$



$$\dot{x} = A_p x + B_p u$$

$$y = C_p x + D_p u$$

$$\Delta = \text{diag}\{\delta_1 I, \cdots, \delta_S I\}$$

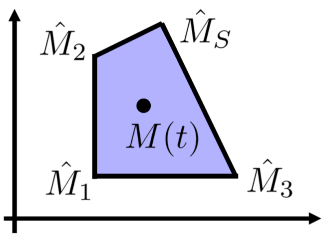
$$A_p = A + \sum_{i=1}^{S} \delta_i \hat{A}_i \quad B_p = B + \sum_{i=1}^{S} \delta_i \hat{B}_i$$

$$C_p = C + \sum_{i=1}^{S} \delta_i \hat{C}_i \quad D_p = D + \sum_{i=1}^{S} \delta_i \hat{D}_i$$

Linear parameter varying (LPV) system

Polytopic-type system

Affine parameter-dependent system



⇒ Gain Scheduled H_∞ Problem

(Ref 1, p. 292)

Diagonal Uncertainty

- **Allowed Structure** $\boldsymbol{\Delta} = \text{diag}\{\delta_1 I_{r1}, \cdots, \delta_S I_{rS}, \Delta_1, \cdots, \Delta_F\}$

$$\boldsymbol{\Delta} = \left[\begin{array}{ccc|ccc} \delta_1 I_{r1} & & & & & \\ & \ddots & & & 0 & \\ & & \delta_S I_{rS} & & & \\ \hline & & & \Delta_1 & & \\ & 0 & & & \ddots & \\ & & & & & \Delta_F \end{array}\right]$$

Parametric Uncertainties:

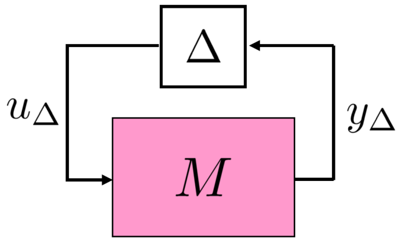
$$\delta_i \in \mathcal{R}, \; i = 1, \cdots, S$$

Nonparametric Uncertainties:

$$\Delta_j \in \mathcal{C}^{m_j \times m_j}, \; j = 1, \cdots, F$$

- **Allowed Perturbations** $\forall \Delta \in B_{\boldsymbol{\Delta}}$

$$B_{\boldsymbol{\Delta}} = \{\Delta \in \boldsymbol{\Delta} \mid \|\Delta\|_\infty \le 1\}$$

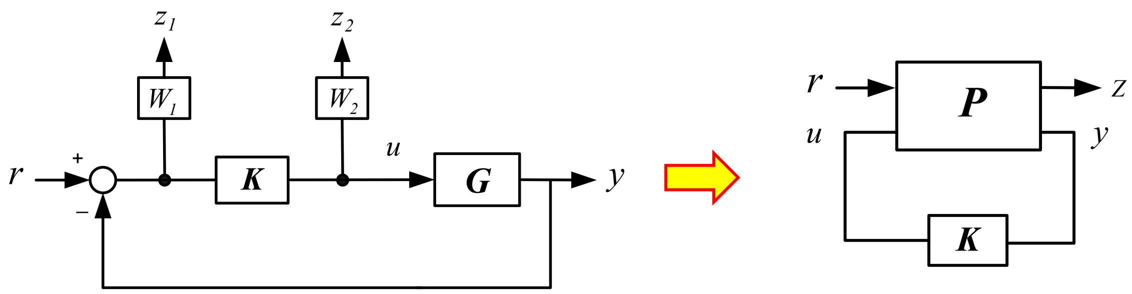


(Ref 1, pp. 289, 296, 300)

Project: Report 6

Consider your dynamic system :

Using uncertainty and performance weighing functions (W2 and W1) in the previous project steps, find the lower linear fractional transformation (LLFT), according to the following block diagrams.



Deadline: The day before next Meeting

Please only use this email address: bevranih18@gmail.com

Thank You!