Robust Control Systems

H∞ Control Design and Order Reduction

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Contents

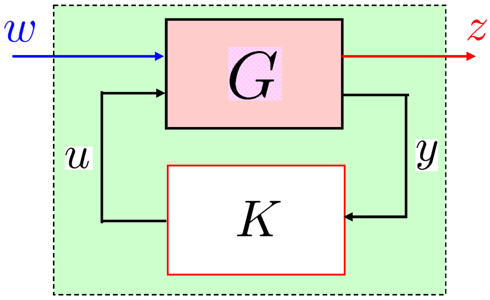
Reference

1. S. Skogestadand I. Postlethwaite, **Multivariable Feedback Control; Analysis and Design**, Second Edition, Wiley, 2005.

2. M. Fujita, **Lecture Notes on Feedback Control Systems**, Tokyo Institute of Technology, 2019.

3. M. Hirata, **Practical Robust Control**, CORONA Press , 2017 (In Japanese).

General Control Problem Formulation



u : control inputs

y : measured (or sensor) outputs

w : exogenous inputs (disturbance and commands, etc.)

z : regulated outputs

○ **Generalized Plant:**

$$\begin{bmatrix} z \\ y \end{bmatrix} = G(s) \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

○ **Closed-loop Transfer Function (LFT):**

$$z = F_l(G, K)w$$
$$F_l(G, K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$$

lft(G,K)

(Ref 1, p. 104)

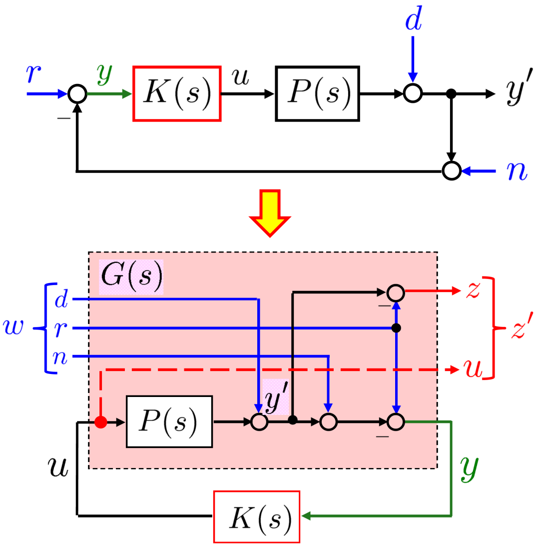
General Control Problem Formulation: Example 1

Exogenous Inputs: $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} d \\ r \\ n \end{bmatrix}$

Controlled Output: $z = y' - r$

Measured Output: $y = r - y' - n$

Control Input: $u = u$



- Building Interconnection

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} y' - r \\ r - y' - n \end{bmatrix} = \begin{bmatrix} Pu + d - r \\ r - Pu - d - n \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} I & -I & 0 & P \\ \hline -I & I & -I & -P \end{array}\right] \begin{bmatrix} d \\ r \\ n \\ u \end{bmatrix} = G \begin{bmatrix} w \\ u \end{bmatrix}$$

- Generalized Plant $G = \left[\begin{array}{ccc|c} I & -I & 0 & P \\ \hline -I & I & -I & -P \end{array}\right]$ $\quad z' = \begin{bmatrix} y' - r \\ u \end{bmatrix}$

(Ref 1, p. 105)

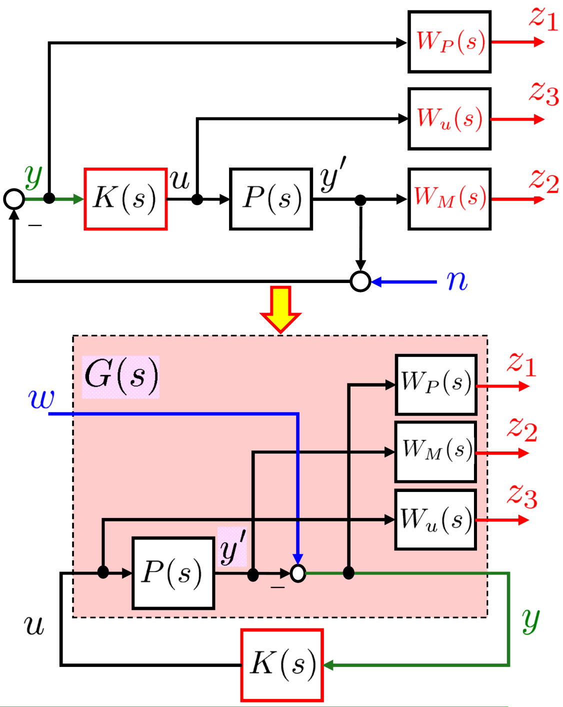
General Control Problem Formulation: Example 2

Exogenous Inputs: $w = n$

Regulated Output: $z = \begin{bmatrix} W_P(y' + w) \\ W_M y' \\ W_u u \end{bmatrix}$

Measured Output: $y = -y' - n$

Control Input: $u = u$



- Building Interconnection and Generalized Plant

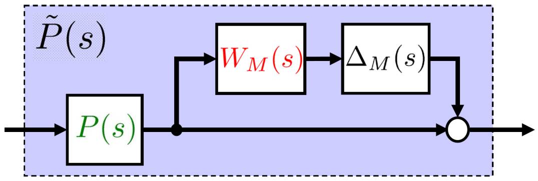
$$\begin{bmatrix} z \\ y \end{bmatrix} = G \begin{bmatrix} w \\ u \end{bmatrix}, \quad G = \left[\begin{array}{c|c} W_P & W_P P \\ 0 & W_M P \\ 0 & W_u \\ \hline -I & -P \end{array}\right]$$

(Ref 1, p. 107)

Example: Spinning Satellite

Nominal Model

$$P(s) = \begin{bmatrix} \frac{s-100}{s^2+100} & \frac{10s+10}{s^2+100} \\ \frac{-10s-10}{s^2+100} & \frac{s-100}{s^2+100} \end{bmatrix}$$



- **Multiplicative (Output) Uncertainty**

$$\Pi_0 = \{\tilde{P}(s) |\ \tilde{P}(s) = (I + \Delta_M(s) W_M(s)) P(s),\ \|\Delta_M\|_\infty \le 1\}$$

- **Uncertainty Weight**

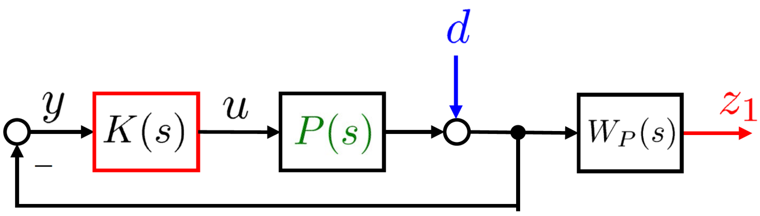
$$W_M(s) = w_M(s) I_2,\ w_M(s) = \frac{0.045s + 0.4}{0.018s + 1} \quad (\tau = 0.045,\ r_0 = 0.4,\ r_\infty = 2.5)$$

$1/\tau = 22.2$

- **Performance Weight**

$$W_P(s) = w_p(s) I_2\ ,\ w_p(s) = \frac{0.5s + 2}{s + 0.02}$$

$(\omega_b = 2,\ A = 0.01,\ M_S = 2)$



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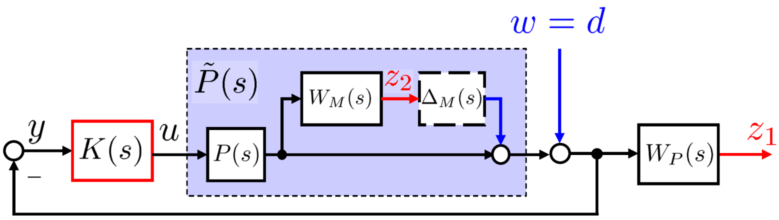
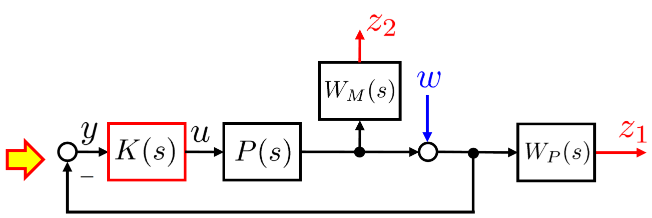
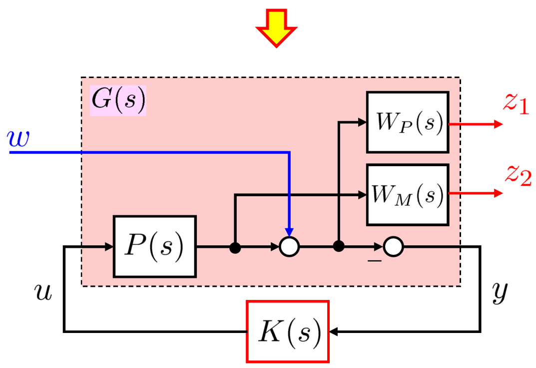




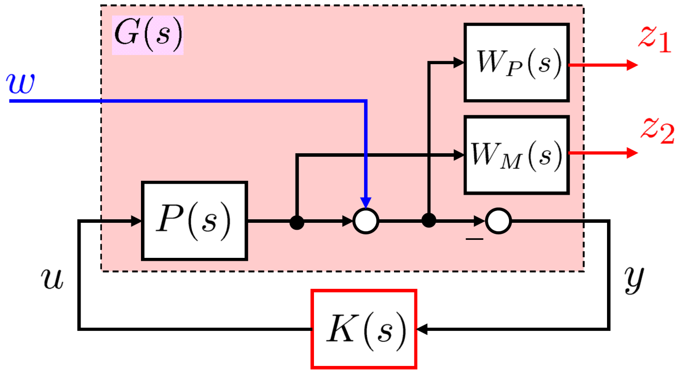


$$\begin{bmatrix} z \\ y \end{bmatrix} = G(s) \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

Building Interconnection Using MATLAB

MATLAB Command

```
%Generalized Plant%
systemnames = ‘Pnom WP WM';
inputvar = '[w(2);u(2)]';
outputvar = '[WP;WM;-w-Pnom]';
input_to_Pnom= '[u]';
input_to_WP = '[w+Pnom]';
input_to_WM = ‘[Pnom]';
G = sysic;
```



$H\infty$ Control Problem

- **$H\infty$ Optimal Control Problem:** Find all stabilizing controllers which minimize

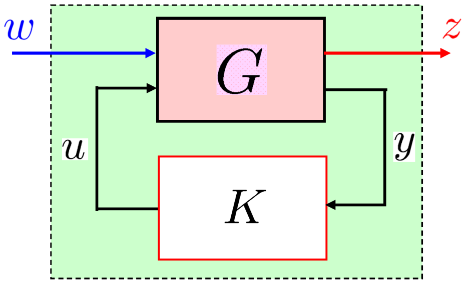
$$\|F_l(G,K)\|_\infty = \max_\omega \bar{\sigma}(F_l(G,K)(j\omega))$$

- **$H\infty$ Sub-optimal Control Problem:**

  Given $\gamma > \gamma_{min}$, find all stabilizing controllers K such that

$$\|F_l(G,K)\|_\infty < \gamma \qquad \gamma \text{ -iteration}$$

$$z = F_l(G,K)w$$

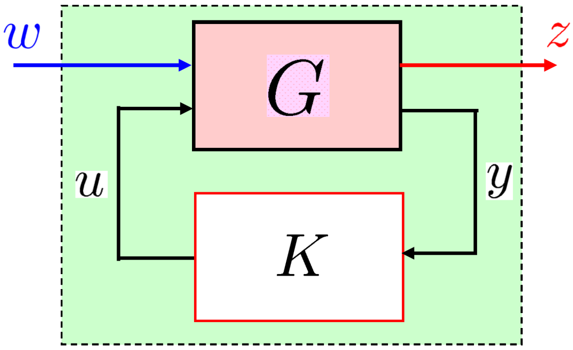


(Ref 1, pp. 104-114)

$H\infty$ Control Problem: Some Applications

- Sensitivity Minimization Problem
- Robust Stabilization Problem
- Mixed Sensitivity Problem
- LQG Type Control Problem
- Feedforward Problem
- Estimation Problem

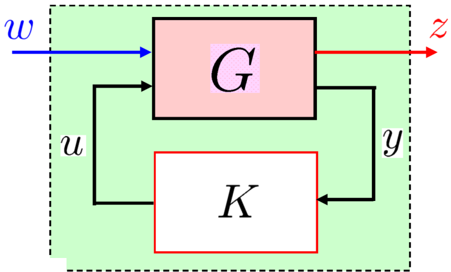
$$z = F_l(G,K)w$$



$H\infty$ Control: State Space Approach

$$\begin{bmatrix} z \\ y \end{bmatrix} = G(s)\begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}\begin{bmatrix} w \\ u \end{bmatrix}$$

$$u = K(s)y$$



- **Generalized Plant**

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \qquad \begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w + D_{22} u \end{aligned}$$

- **Closed-loop Transfer Function (LFT):** $F_l(G,K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$

- **$H\infty$ Control Problem:** Given $\gamma > \gamma_{min}$, find all stabilizing controllers K such that

$$\|F_l(G,K)\|_\infty < \gamma$$

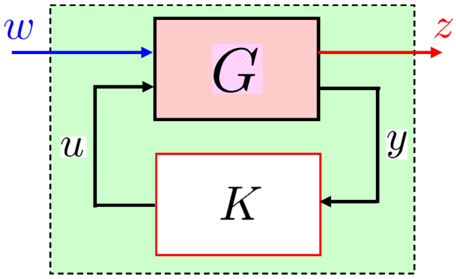
(Ref 1, p. 357)

A Simplified $H\infty$ Control Problem

- **Generalized Plant**

$$\dot{x} = Ax + B_1 w + B_2 u$$
$$z = C_1 x + D_{12} u$$
$$y = C_2 x + D_{21} w$$

$$\Rightarrow \quad G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]$$



- **Assumptions**

(A1) (A, B_2) is stabilizable and (C_2, A) is detectable

(A2) (A, B_1) is controllable and (C_1, A) is observable

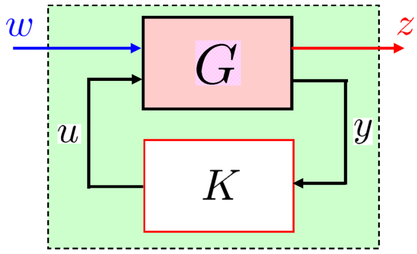
(A3) $D_{12}^T [C_1 \;\; D_{12}] = [0 \quad I]$ and $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}$

(Ref 1, p. 353)

Bounded Real Lemma

- **For $\gamma > 0$, $F_l = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right]$, the following two conditions are equivalent:**

$$z = F_l(G, K)w$$



(i) $\|F_l\|_\infty < \gamma$

(ii) There exists a $P_c > 0$ such that

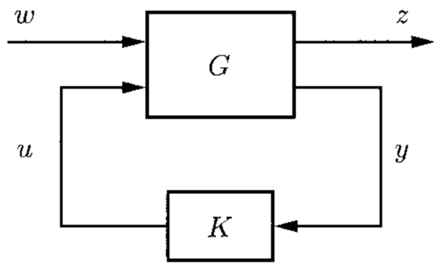
$$P_c A_c + A_c^T P_c + \gamma^{-2} P_c B_c B_c^T P_c + C_c^T C_c = 0$$

and $A_c + \gamma^{-2} B_c B_c^T P_c$ has no eigenvalues on the imaginary axis.

K. Zhou with J.C. Doyle, Essentials of Robust Control, Prentice Hall, 1998

Review: LFT and ∞-Norm

$$\begin{bmatrix} z \\ y \end{bmatrix} = G\begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}\begin{bmatrix} w \\ u \end{bmatrix}$$



$z = G_{zw}w$ ⇨

$$G_{zw} = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} = G_{11} + G_{12}(I - KG_{22})^{-1}KG_{21}$$

$$\| G_{zw}(s) \|_\infty := \sup_\omega \overline{\sigma}\{G_{zw}(j\omega)\}$$

$$\| G_{zw} \|_\infty = \sup_{w\neq 0} \frac{\sqrt{\int_0^\infty z^T(t)z(t)dt}}{\sqrt{\int_0^\infty w^T(t)w(t)dt}} = \sup_{w\neq 0}\frac{\| z \|_2}{\| w \|_2}$$

$$\overline{\sigma}(M) = \sqrt{\lambda_{\max}(M^*M)}$$

- **∞-Norm Characteristics:**

① $\| G \|_\infty \geq 0$

② $\| G \|_\infty = 0 \Leftrightarrow G = 0$

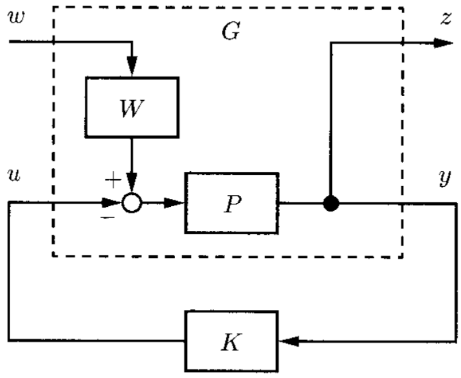
③ $\| \alpha G \|_\infty = |\alpha| \; \| G \|_\infty, \quad \alpha \in \mathcal{C}$

④ $\| G + H \|_\infty \leq \| G \|_\infty + \| H \|_\infty$

⑤ $\| GH \|_\infty \leq \| G \|_\infty \| H \|_\infty$

Review: Generalized Plant

$$G_{zw} = \frac{P}{1+PK}W \qquad \begin{bmatrix} z \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} PW & -P \\ PW & -P \end{bmatrix}}_{G}\begin{bmatrix} w \\ u \end{bmatrix}$$



$P = (A_p, B_p, C_p, 0)$ ⇨
$$\begin{cases} \dot{x}_p(t) = A_p x_p(t) + B_p(-u(t) + y_w(t)) \\ z(t) = C_p x_p(t) \\ y(t) = C_p x_p(t) \end{cases}$$

$W = (A_w, B_w, C_w, D_w)$ ⇨
$$\begin{cases} \dot{x}_w(t) = A_w x_w(t) + B_w w(t) \\ y_w(t) = C_w x_w(t) + D_w w(t) \end{cases}$$

⇨
$$\dot{x}_p(t) = A_p x_p(t) + B_p C_w x_w(t) + B_p D_w w(t) - B_p u(t)$$

⇨

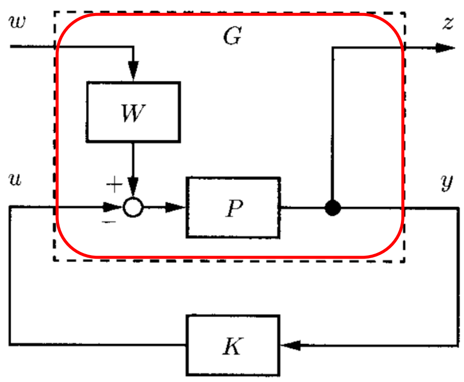
$$\begin{bmatrix} \dot{x}_p(t) \\ \dot{x}_w(t) \end{bmatrix} = \begin{bmatrix} A_p & B_pC_w \\ 0 & A_w \end{bmatrix}\begin{bmatrix} x_p(t) \\ x_w(t) \end{bmatrix} + \begin{bmatrix} B_pD_w & -B_p \\ B_w & 0 \end{bmatrix}\begin{bmatrix} w(t) \\ u(t) \end{bmatrix}$$

$$\begin{bmatrix} z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} C_p & 0 \\ C_p & 0 \end{bmatrix}\begin{bmatrix} x_p(t) \\ x_w(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} w(t) \\ u(t) \end{bmatrix}$$

Review: Generalized Plant Building in MATLAB

- **Example:** $P = \dfrac{10}{s+1}, \quad W = \dfrac{1}{s+5}$

```
%% Generalized Plant Building using sysic
% Introduce the blocks
s = tf('s');
P = ss(10/(s+1)); % Convert to state-space
W = ss(1/(s+5)); % Convert to state-space
% Defining a generalized plant with sysic
systemnames = 'P W';
inputvar = '[w; u]'; % Input signals
outputvar = '[P; P]'; % Output signals
input_to_P = '[W - u]'; % Input to P
input_to_W = '[w]'; % Input to W
G = sysic; % Building G
```



- **You can use *connect* function instead of *sysic* in MATLAB**

A Discussion on Required Assumption in Standard H∞ Control

- **It has been already emphasized that in the given generalized plant the D_{12} and D_{21} must be full rank (in row and column perspective, respectively).**

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1w(t) + B_2u(t) \\ z(t) = C_1x(t) + D_{11}w(t) + D_{12}u(t) \\ y(t) = C_2x(t) + D_{21}w(t) \end{cases} \Rightarrow \begin{cases} \dot{x}(t) = Ax(t) + \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \end{bmatrix} \\ \begin{bmatrix} z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x(t) + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & O \end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \end{bmatrix} \end{cases}$$

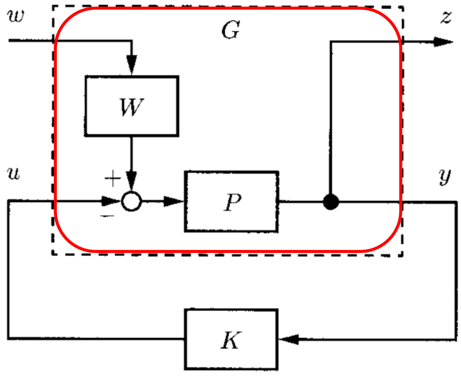
$$\Rightarrow \quad G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & O \end{array}\right]$$

- **What should we do if this requirement is not satisfied?**

A Discussion on Required Assumption in Standard H∞ Control

Example: Consider the former system

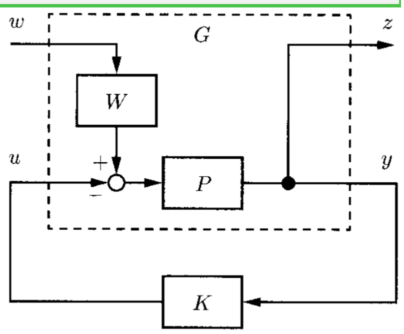
$$\begin{cases} \begin{bmatrix} \dot{x}_p(t) \\ \dot{x}_w(t) \end{bmatrix} = \begin{bmatrix} A_p & B_pC_w \\ 0 & A_w \end{bmatrix} \begin{bmatrix} x_p(t) \\ x_w(t) \end{bmatrix} + \begin{bmatrix} B_pD_w & -B_p \\ B_w & 0 \end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \end{bmatrix} \\ \begin{bmatrix} z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} C_p & 0 \\ C_p & 0 \end{bmatrix} \begin{bmatrix} x_p(t) \\ x_w(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \end{bmatrix} \end{cases}$$



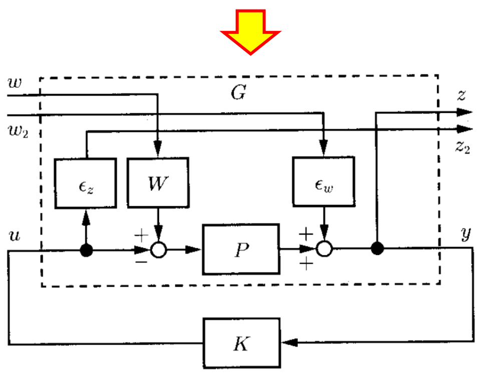
Solution: The problem can be solved by adding additional input and/or output with a too small gain(s).

A Discussion on Required Assumption in Standard H∞ Control

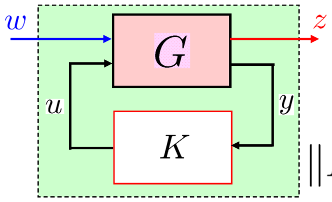
$$\begin{cases} \begin{bmatrix} \dot{x}_p(t) \\ \dot{x}_w(t) \end{bmatrix} = \begin{bmatrix} A_p & B_pC_w \\ 0 & A_w \end{bmatrix} \begin{bmatrix} x_p(t) \\ x_w(t) \end{bmatrix} + \begin{bmatrix} B_pD_w & -B_p \\ B_w & 0 \end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \end{bmatrix} \\ \begin{bmatrix} z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} C_p & 0 \\ C_p & 0 \end{bmatrix} \begin{bmatrix} x_p(t) \\ x_w(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \end{bmatrix} \end{cases}$$



$$\begin{cases} \begin{bmatrix} \dot{x}_p(t) \\ \dot{x}_w(t) \end{bmatrix} = \begin{bmatrix} A_p & B_pC_w \\ 0 & A_w \end{bmatrix} \begin{bmatrix} x_p(t) \\ x_w(t) \end{bmatrix} + \left[\begin{array}{cc:c} B_pD_w & 0 & -B_p \\ B_w & 0 & 0 \end{array}\right] \left[\begin{array}{c} w(t) \\ w_2(t) \\ \hdashline u(t) \end{array}\right] \\ \left[\begin{array}{c} z(t) \\ z_2(t) \\ \hdashline y(t) \end{array}\right] = \left[\begin{array}{cc} C_p & 0 \\ 0 & 0 \\ \hdashline C_p & 0 \end{array}\right] \begin{bmatrix} x_p(t) \\ x_w(t) \end{bmatrix} + \left[\begin{array}{cc:c} 0 & 0 & 0 \\ 0 & 0 & \epsilon_z \\ \hdashline 0 & \epsilon_w & 0 \end{array}\right] \left[\begin{array}{c} w(t) \\ w_2(t) \\ \hdashline u(t) \end{array}\right] \end{cases}$$



General $H\infty$ Control Solutions Using MATLAB



$$\|F_l(G,K)\|_\infty < \gamma$$

[k, cl, gam, info] = **hinfsyn**(p, nmeas, ncon, key1, value1, key2, value2, ...)

input argument

- p — generalized plant
- nmeas — number of measurement outputs
- ncon — number of control inputs

output argument

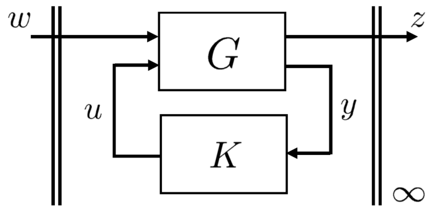
- k — LTI controller
- cl — closed loop system which consists of K and G
- gam — H_∞ norm of closed loop system
- info — information of output results

Key setting

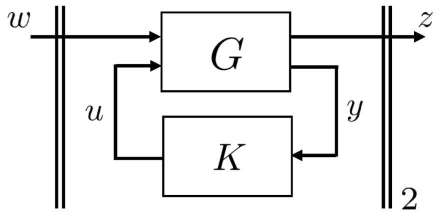
- Gmax — upper limit of Gam
- Gmin — lower limit of Gam
- Tolgam — relative error of Gam
- So — frequency at which entropy is assessed
- Method — Ric : Ricatti solution; Lmi : LMI solution; Maxe : max entropy solution
- Display — Off : not show setting process; On : show setting process

All Robust Control Solutions

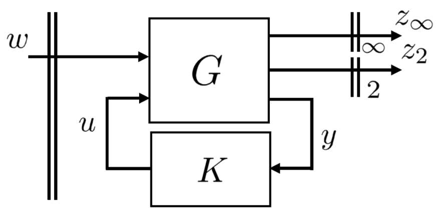
H∞ controller synthesis
hinfsyn



H2 controller synthesis
h2syn

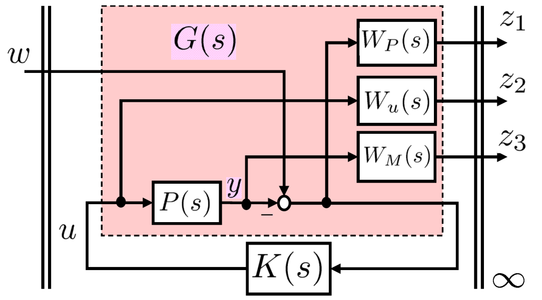


Mixed H2/H∞ controller synthesis
h2hinfsyn

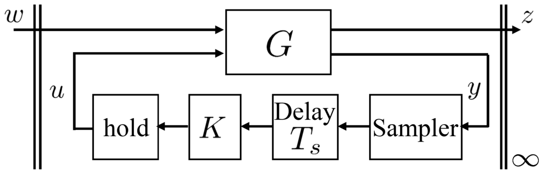


Robust Control Solutions

H∞ mixed sensitivity
controller synthesis
mixsyn



Sample-data
H∞ controller synthesis
sdhinfsyn

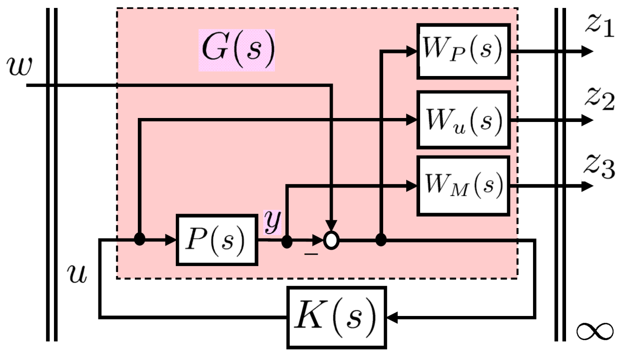


Robust Control Solutions

H∞ loop shaping controller synthesis
loopsyn

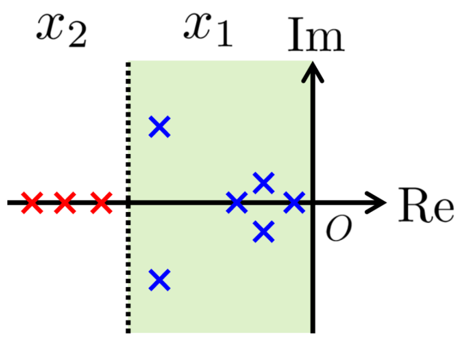
Sample-data H∞ irreducible decomposition
controller synthesis
ncfsyn

μ-controller synthesis
dksyn



Model Reduction

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \Rightarrow \begin{cases} \dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1 u \\ \dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2 u \\ y = C_1 x_1 + C_2 x_2 + Du \end{cases}$$



G : stable $x \in \mathbb{R}^n$ $\quad x^T = [\ x_1^T\ x_2^T\ ]^T,\ \ x_1 \in \mathbb{R}^k$

$A = \text{diag}(\lambda_1, \cdots, \lambda_n)\ ,\ \ |\lambda_1| < |\lambda_2| < \cdots < |\lambda_n|\ ,$
$B = [b_i^T]\ ,\ C = [c_i]$ ⇒ x_2 contains the fast modes

Model Reduction

○ **Modal Truncation** $(x_2 \to 0)$

MATLAB Command
[Gs,Gf] = slowfast(G,ns);

$$G_a = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right] \quad \begin{array}{l} G(j\infty) = G_a(j\infty) = D \\ G - G_a = \sum_{i=k+1}^{n} \frac{c_i b_i^T}{s - \lambda_i} \end{array} \Rightarrow \|G - G_a\|_\infty \leq \sum_{i=k+1}^{n} \frac{\bar{\sigma}(c_i b_i^T)}{|\text{Re}(\lambda_i)|}$$

Model Error

○ **Residualization** $(\dot{x}_2 \to 0)$

MATLAB Command
[SysG1,SysG2] = modreal(G,cut);

$$G_a = \left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array}\right] \begin{cases} A_r = A_{11} - A_{12}A_{22}^{-1}A_{21} \\ B_r = B_1 - A_{12}A_{22}^{-1}B_2 \\ C_r = C_1 - C_2 A_{22}^{-1}A_{21} \\ D_r = D - C_2 A_{22}^{-1} B_2 \end{cases} \qquad G(0) = G_a(0)$$

More Model Reduction Methods

- **Balanced Realizations**

MATLAB Command
```
[Sysb,g] = balreal(G);
```

- **Balanced Truncation/Residualization**

MATLAB Command
```
[GRED,info] = balancmr(G,order);
```

- **Optimal Hankel Norm Approximation**

MATLAB Command
```
GRED = hankelmr(G,order);
```

- **Reduction of unstable models**

  1. Stable part model reduction

MATLAB Command
```
[Gs,Gus,m] = stabproj(G);
```

  2. Coprime factor model reduction

MATLAB Command
```
GRED = ncfmr(G,order);
```


Model Reduction Methods in MATLAB

MATLAB Commands (Model Simplification)

Control System Toolbox	Robust Control Toolbox
balred	balancmr
modred	bstmr
sminreal	dcgainmr
minreal	hankelmr
balreal	modreal
hsvd	ncfmr
hsvplot	schurmr
balredOptions	slowfast
hsvdOptions	reduce
	hankelsv

Thank You!