Robust Control Systems

H∞ Robust Control Design: A MIMO System Example

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Contents

Reference

1. S. Skogestadand I. Postlethwaite, **Multivariable Feedback Control; Analysis and Design**, Second Edition, Wiley, 2005.

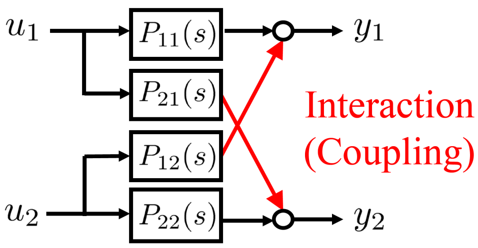
2. M. Fujita, **Lecture Notes on Feedback Control Systems**, Tokyo Institute of Technology, 2019.

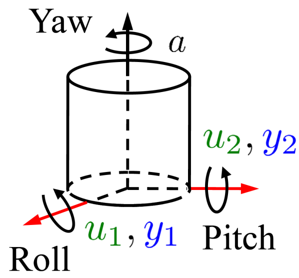
3. R. Smith, **Lecture Notes on Control Systems**, ETH Zurich, 2020.

Nominal Plant Model

- **Transfer Function Matrix**

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \begin{bmatrix} \frac{s-100}{s^2+100} & \frac{10s+10}{s^2+100} \\ \frac{-10s-10}{s^2+100} & \frac{s-100}{s^2+100} \end{bmatrix}$$





MATLAB Command

```
N = { [1 -100],[10 10];[-10 -10],[1 -100] } ;
D = [1 0 100] ;
Pnom = tf(N,D) ;
Pnom = ss(Pnom,'min') ;
```

- **State Space Representation**

$$P(s) = C(sI - A)^{-1}B + D$$

$$P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \qquad \dot{x} = Ax + Bu, \quad y = Cx + Du$$

$$A = \begin{bmatrix} 0 & 10 \\ -10 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 10 \\ -10 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

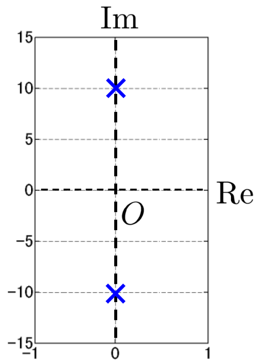
MATLAB Command

```
sysA = [0 10; -10 0]; sysB = eye(2);
sysC = [1 10; -10 1]; sysD = zeros(2);
Pnom = ss(sysA, sysB, sysC, sysD);
```

Characteristics of Nominal Plant Model

- **Poles:**

$p = \pm 10j$ Unstable Poles
Vibrational System

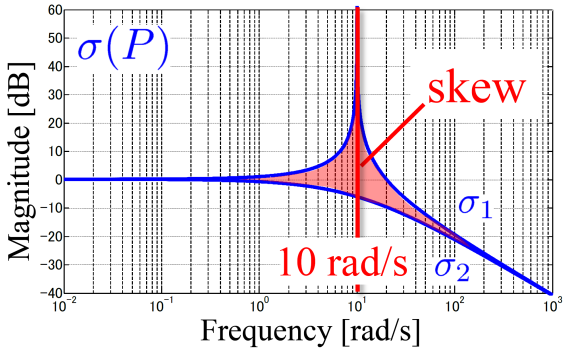


MATLAB Command
pole(Pnom) tzero(Pnom)

MATLAB Command
figure pzmap(Pnom)

MATLAB Command
sigma(Pnom)

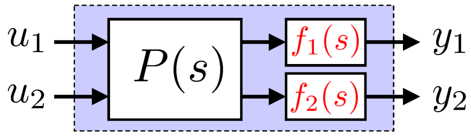
- **Frequency Response (σ -plot)**



Plant Model with Output Uncertainty

- **Uncertain Plant Model**

$$\tilde{P}(s) = \begin{bmatrix} f_1(s) & 0 \\ 0 & f_2(s) \end{bmatrix} P(s)$$



$$f_i(s) = k_i \frac{-\frac{\theta_i}{2}s + 1}{\frac{\theta_i}{2}s + 1}, \quad i = 1, 2$$

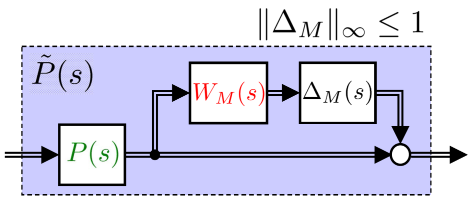
Gain Margin: $0.8 \le k_i \le 1.2$
Delay Margin: $0 \le \theta_i \le 0.02$

- **Multiplicative (Output) Uncertainty**

$$\Pi_0 = \{\tilde{P}(s) |\ \tilde{P}(s) = (I + \Delta_M(s) W_M(s)) P(s),\ \|\Delta_M\|_\infty \le 1\}$$

Uncertainty Weight:

$$W_M(s) = w_M(s) I_2\ , \quad w_M(s) = \frac{\tau s + r_0}{\frac{\tau}{r_\infty} s + 1}$$



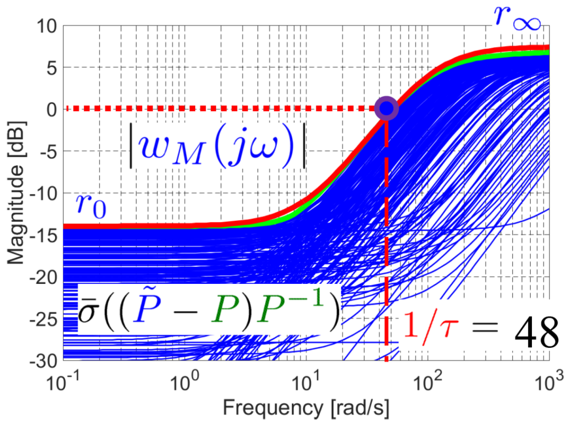
Uncertainty Weight

$$w_M(s) = \frac{\tau s + r_0}{\frac{\tau}{r_\infty} s + 1} \quad \Rightarrow \quad w_M(s) = \frac{0.021s + 0.2}{0.0091s + 1}$$

$\tau = 0.021$,
$1/\tau = 48$ rad/s
$r_0 = 0.2$,
$r_\infty = 2.3$

MATLAB Command

```
r0 = 0.2; rinf = 2.3; tau = 0.021;
wM = tf([tau r0], [tau/rinf 1]);
WM = eye(2)*wM;
WM = ss(WM) ;
```



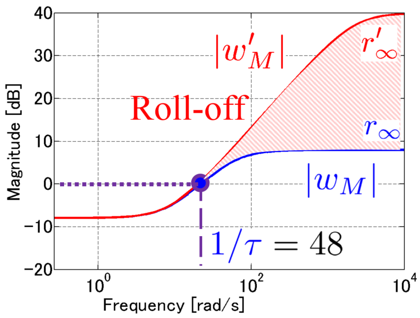
Update Uncertainty Weight

Uncertain Factors =
Gain, Time Delay + Unmodeled Dynamics (High Freq.)

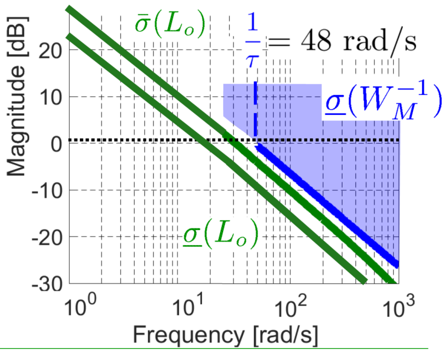
$$w_M(s) = \frac{0.021s + 0.2}{0.0091s + 1}$$

$r_\infty = 2.3 \rightarrow 100$

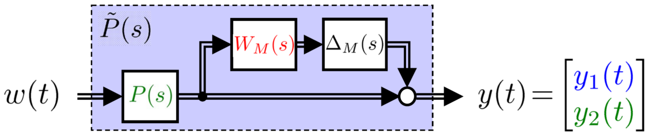
$$w'_M(s) = \frac{0.021s + 0.2}{2.1 \times 10^{-4}s + 1}$$

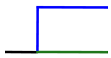


Target Loop (Roll-off)

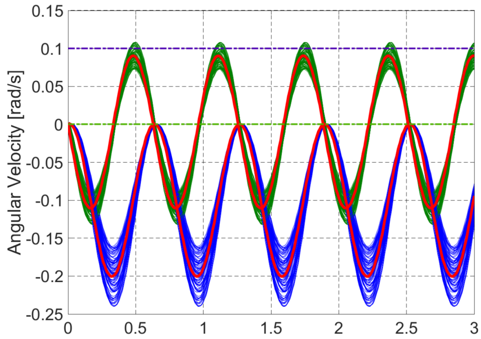


Time Responses for Uncertain Plant Model





$$w(t) = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}$$



MATLAB Command

```
time = 0:0.01:3;
step_ref = ones(1,length(time));
ref = [0.1*step_ref'; 0*step_ref'];
```

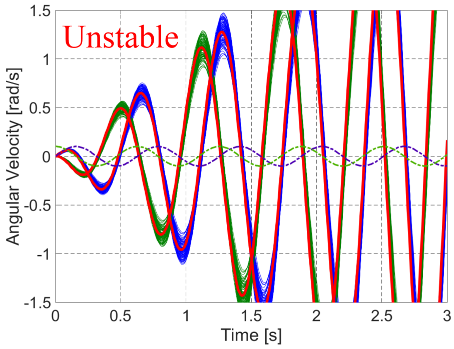
Reference
Nominal Model
Uncertain Model

Reference Signal

```
figure
plot(time,ref,'g-.')
```

$$w(t) = \begin{bmatrix} 0.1\sin(\omega t) \\ 0.1\cos(\omega t) \end{bmatrix}$$

$$\omega = 10 \ \text{rad/s}$$



Performance Weight

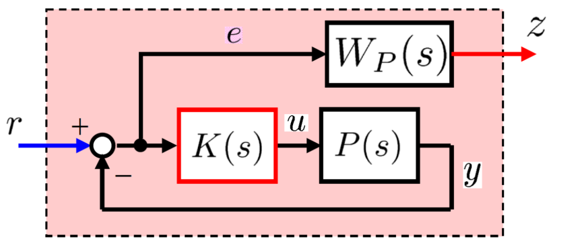
$$W_P(s) = w_p(s) I_2,$$

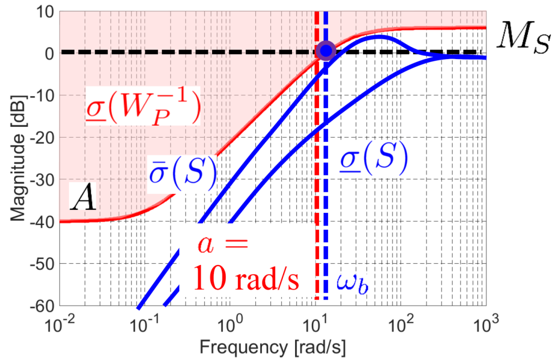
$$w_p(s) = \frac{\frac{1}{M_S}s + \omega_b}{s + \omega_b A} = \frac{0.5s + 11.5}{s + 0.115}$$

$$\omega_b = 11.5 \ (\geq 1.15|p|$$
$$M_s = 2, A = 0.01$$

MATLAB Command

```
Ms = 2; A = 1e-2; wb = 11.5;
wP = tf([1/Ms wb], [1 wb*A]);
WP = eye(2)*wP; WP = ss(WP) ;
```





Update Performance Weight

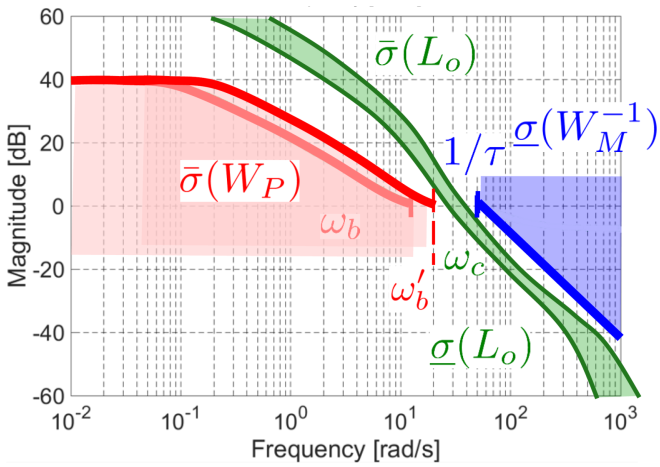
$$w_p(s) = \frac{\frac{1}{M_S}s + \omega_b}{s + \omega_b A} = \frac{0.5s + 11.5}{s + 0.115}$$

$\omega_b = 11.5 \rightarrow 20$

$M_S = 2 \rightarrow 8$ (Trade-off)

$A = 0.01 > 0$

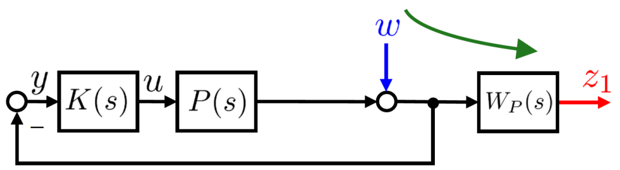
$$w_p'(s) = \frac{0.125s + 20}{s + 0.2}$$



Control Problem Formulation

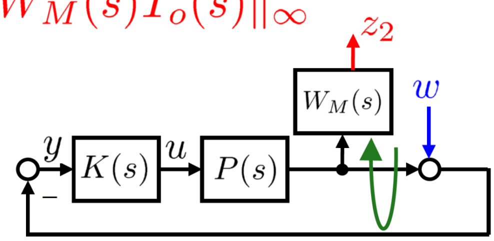
Nominal Performance

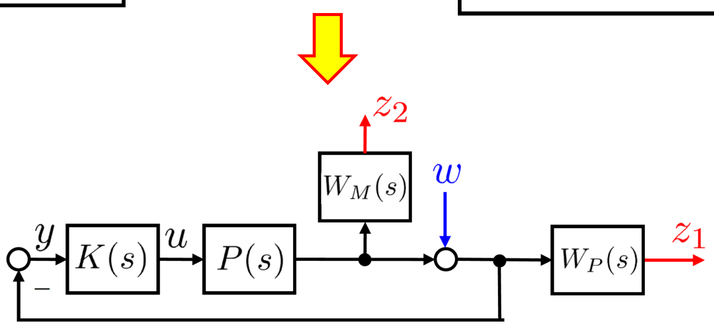
$\|W_P(s)S_o(s)\|_\infty$



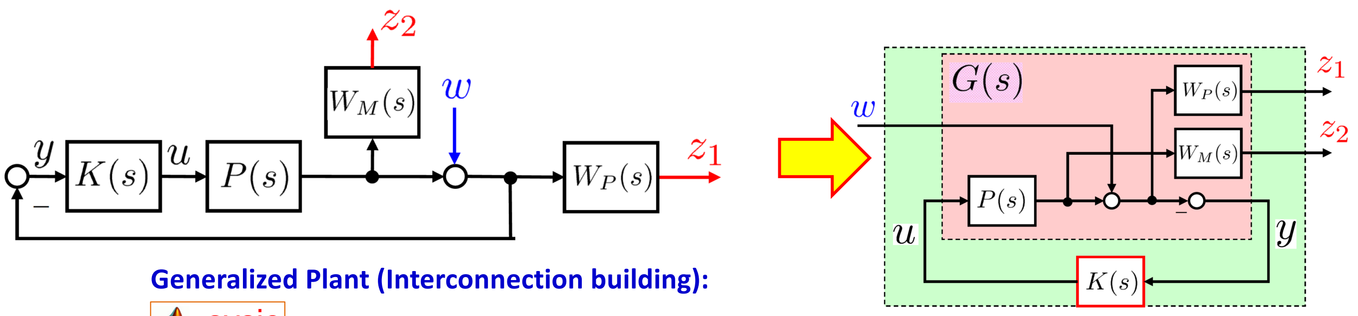
Robust Stability

$\|W_M(s)T_o(s)\|_\infty$





Continue



Generalized Plant (Interconnection building):

sysic

MATLAB Command

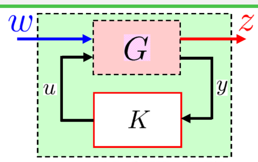
```
%Generalized Plant%
systemnames = 'Pnom WP WM';
inputvar = '[w(2);u(2)]';
outputvar = '[WP;WM;-w-Pnom]';
input_to_Pnom= '[u]';
input_to_WP = '[w+Pnom]';
input_to_WM = ' [Pnom]';
G = sysic;
```

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} W_P(s)S_o(s) \\ -W_M(s)T_o(s) \end{bmatrix} w$$

$$= F_l(G, K)$$

Mixed Sensitivity

H∞ Optimal Controller



$z = F_l(G, K)w$

H∞ Control Problem hinfsyn

Given $\gamma > \gamma_{min}$, find all stabilizing controllers K such that $\|F_l(G, K)\|_\infty < \gamma$.

[K, CL, gam, info] = hinfsyn(G, nmeas, ncon, key1, value1, key2, value2, …)

Input arguments

- **G**: generalized plant
- **nmeas**: number of control inputs
- **ncon**: number of measurement outputs

Output arguments

- **K**: LTI controller
- **CL**: closed loop system which consists of and
- **gam**: H∞ norm of closed loop system
- **info**: information of output results

key settings

- **Gmax**: upper limit of gam(=Inf)
- **Gmin**: lower limit of gam(=0)
- **Tolgam**: relative error of gam(=0.01)
- **So**: frequency at which entropy is assessed (default=Inf)
- **Display**: off (not show setting process)
  on(show setting process)

MATLAB Command

```
nmeas = 2; ncon = 2;
[Khi,CLhi,ghi,hiinfo] = hinfsyn(G,nmeas,ncon);
ghi
Fhi=loopsens(Pnom,Khi);
```

Method

- **ric**: Ricatti solution (default)
- **lmi**: LMI solution
- **maxe**: max entropy solution

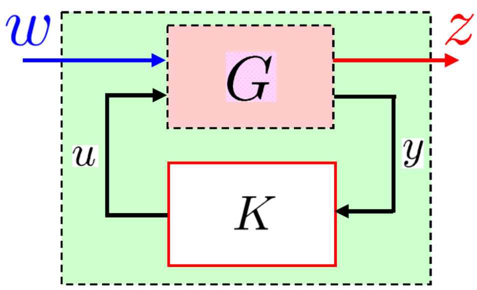
H∞ Optimal Controller

- **Assumptions for Generalized Plant**

(1) (A, B_2) :stabilizable, (C_2, A) :detectable
(2) (A, B_1) :controllable, (C_1, A) :observable

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

hinfsyn



"If the Robust Control toolbox of MATLAB complains, then it probably means that your control problem is not well formulated, and you should think again"

γ -iteration to obtain H∞ Controller

Find K such that $\|F_l(G,K)\|_\infty < \gamma$

- **Check 1** Appropriately sub-optimal (Default settings)

$$\gamma = 0.6719 < 1 \quad (\gamma_{opt} = 0.6650)$$

- **Check 2** $\gamma \to \infty$ (Gmax $= 100$)

$(K \to$ LQG Controller$/H_2)$

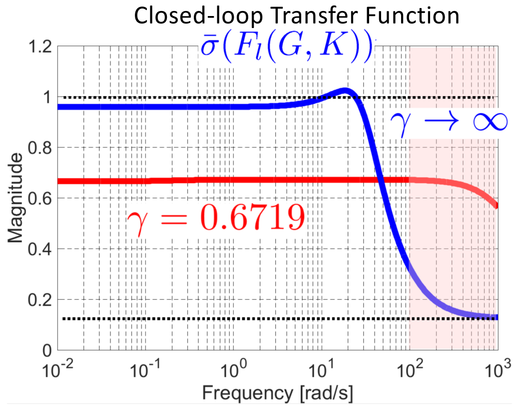
$$\|F_l(G,K)\|_\infty = 1.0232$$

Resetting value of Gamma min based on D_11, D_12, D_21 terms

Test bounds: 0.1250 < gamma <= 0.6719

gamma	hamx_eig	xinf_eig	hamy_eig	yinf_eig	nrho_xy	p/f
0.672	1.6e+01	9.2e-02	5.8e-11	-1.8e-21	0.0036	p
0.398	3.1e-12#	********	5.8e-11	-1.1e-21	********	f
0.535	1.1e+01	-1.4e+00#	5.8e-11	-1.5e-23	0.0002	f
0.604	1.4e+01	-4.1e+00#	5.8e-11	-7.2e-22	0.0004	f
0.638	1.5e+01	-1.1e+01#	5.8e-11	3.1e-22	0.0009	f
0.655	1.6e+01	-3.0e+01#	5.8e-11	-2.6e-21	0.0024	f
0.663	1.6e+01	-1.9e+02#	5.8e-11	-5.5e-21	0.0144	f

Gamma value achieved: 0.6719



"interested" frequency range

MATLAB Command
[SV,w]=sigma(CLhi); figure; semilogx(w,SV)

MATLAB Command
[Khi,CLhi,ghi,hiinfo] = … hinfsyn(G,nmeas,ncon,'Gmax',100,'Gmin', 0.5);

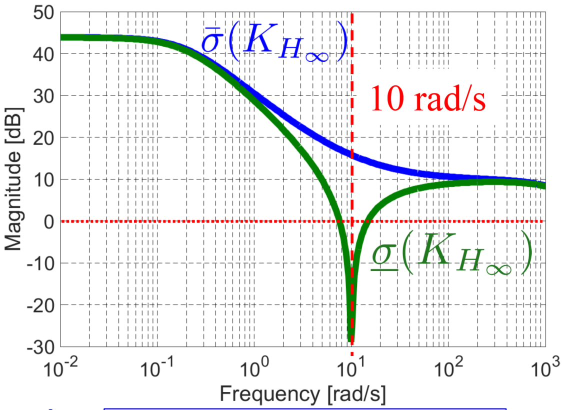
(Ref 1, p. 358)

H∞ Controller

$$K_{H_\infty}(s) = \begin{bmatrix} K_{11}(s) & K_{12}(s) \\ K_{21}(s) & K_{22}(s) \end{bmatrix}$$

Order: 6

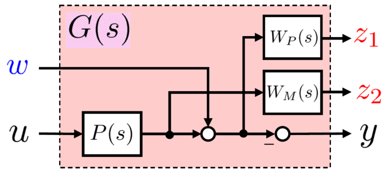
MATLAB Command
figure sigma(Khi)



Poles: $-1.0050 \cdot 10^6 \pm 10j,$ $-1.5011 \cdot 10^3 \pm 1.5493 \times 10^{-4} j$ $-0.1978, -0.1978$

Zeros: $-4762, -4762,$ $\pm 10j$

Generalized Plant : Order 6



$P(s)$: Order 2
$W_M(s)$: Order 2, $W_P(s)$: Order 2

Nominal Plant Model
Poles $\pm 10j$

Mixed Sensitivity Problem
⇒Pole/Zero Cancellations

H∞ Loop-shaping Design

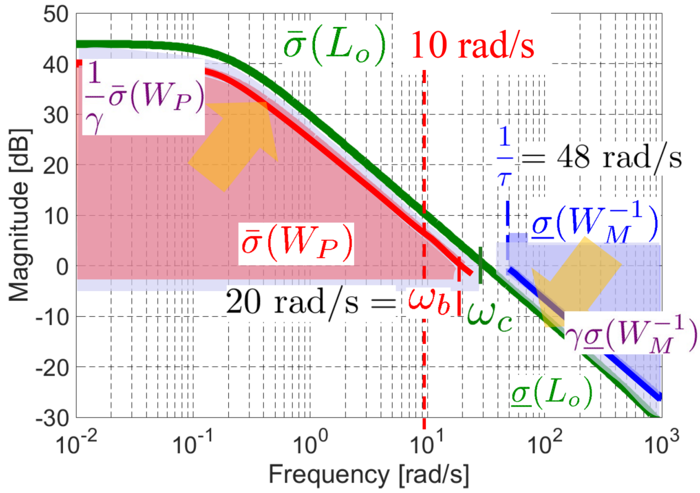
Open-loop Frequency Response Analysis

$$\gamma = 0.6719 < 1$$

(corresponding maximum stability margin)

Loop Transfer Function

MATLAB Command
figure sigma(Fhi.Lo,WP, inv(WM),WP/ghi,ghi*inv(WM))



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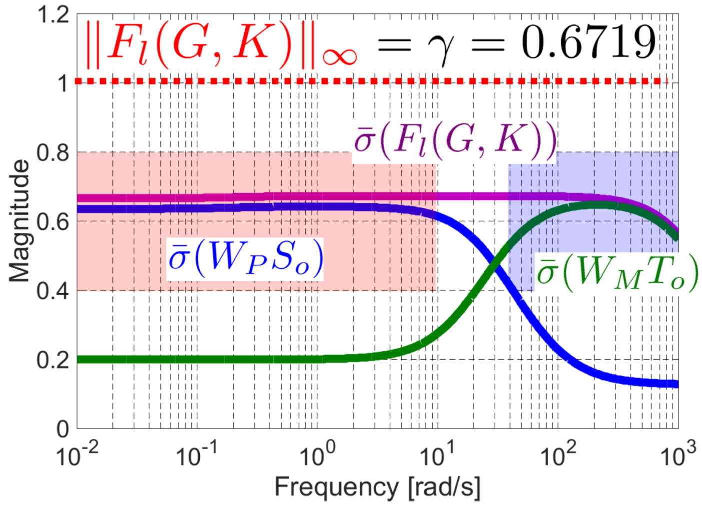


MATLAB Command
[SV,w]=sigma(WP*Fhi.So); figure; semilogx(w,SV) [SV,w]=sigma(WM*Fhi.To); figure; semilogx(w,SV)

Nominal Performance (NP) Test

$\|W_P S_o\|_\infty = 0.6411 < 1$ √

Robust Stability (RS) Test

$\|W_M T_o\|_\infty = 0.6468 < 1$ √

Closed-loop Performance Analysis

○ Nominal Stability (NS)

Poles of $F_l(G,K)$

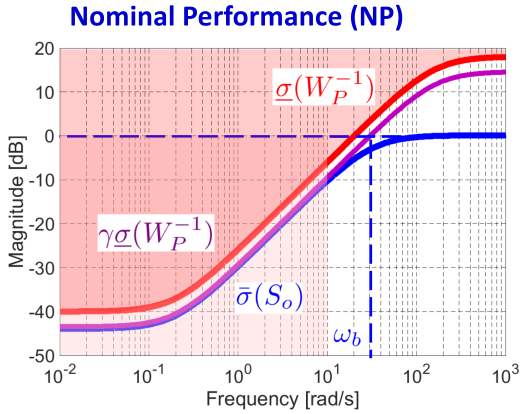
MATLAB Command
pole(CLhi) zero(CLhi) figure; pzmap(CLhi)

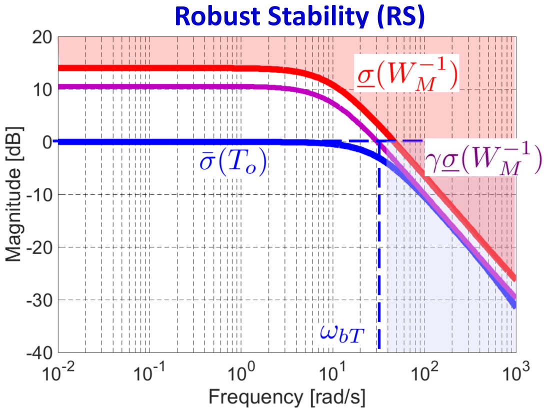
$$p = -0.2, -0.2, -4762, -4762,$$
$$-1.005 \cdot 10^6 \pm 10j, \pm 10j,$$
$$-1.480 \cdot 10^3 \pm 0.0j, -31.63 \pm 0.0j$$

√

Continue

$\gamma = 0.6719 < 1$





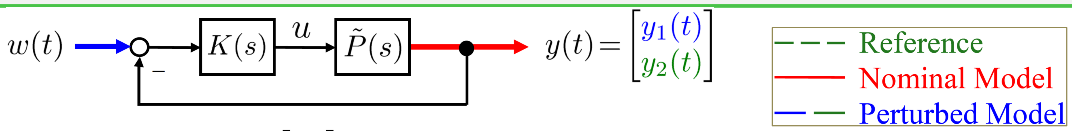
MATLAB Command

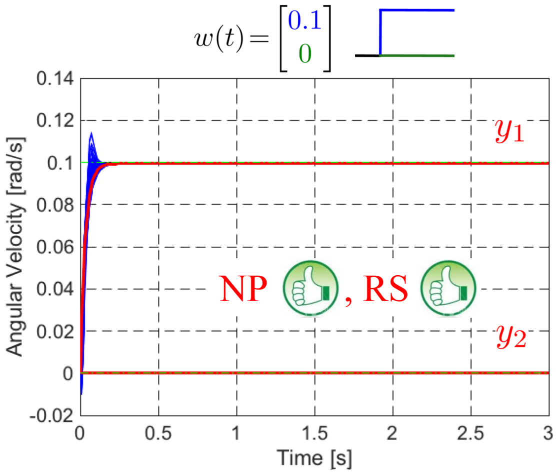
```
figure;
sigma(Fhi.So,inv(WP),ghi*inv(WP))
```

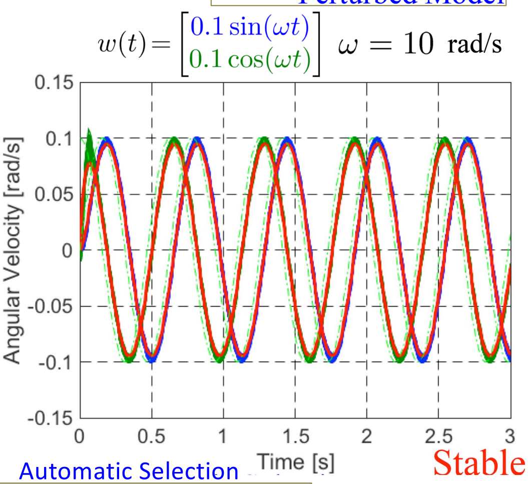
MATLAB Command

```
figure;
sigma(Fhi.To,inv(WM),ghi*inv(WM))
```


Closed-loop Systems Time Responses Analysis







Manual Selection

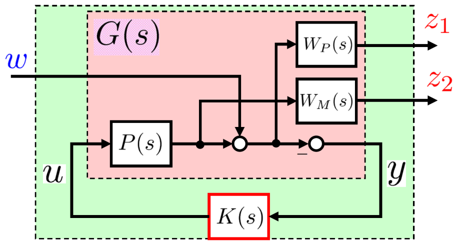
```
[yhi,t] = lsim(F.To,ref,time);
plot(t,yhi(:,1),'b-'); plot(t,yhi(:,2),'g-');
```

Automatic Selection

```
for i = 1 : 100
   [yhi,t] = lsim(Farray.To(:,:,i),ref,time);
   plot(t,yhi(:,1),'b-'); plot(t,yhi(:,2),'g-');
end
```

H∞ Controller Using *mixsyn* function

mixsyn   [K, CL, gam, info] = mixsyn(P, WP, WU, WM, key1, value1, …)

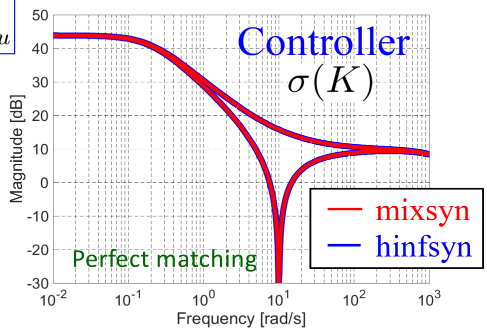


Remark:

1. $P(s)$, $W_P(s)$, $W_u(s)$, $W_M(s)$: proper, $W_P(s)$, $W_u(s)$, $W_M(s)$: stable
   $P(s)$: stabilizable and detectable
2. Each of $W_P(s)$, $W_u(s)$ and $W_M(s)$ must be either
   a) empty (you may simply assign an empty matrix "[]"),
   b) scalar (SISO) or
   c) have respective input dimensions n_y, n_u and n_y where P is n_y-by-n_u

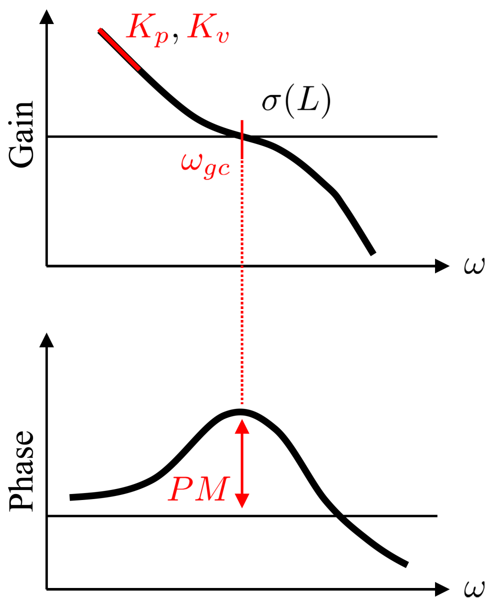
MATLAB Command

```
[Km,CLm,gm,minfo] = mixsyn(Pnom,WP,[],WM);
```

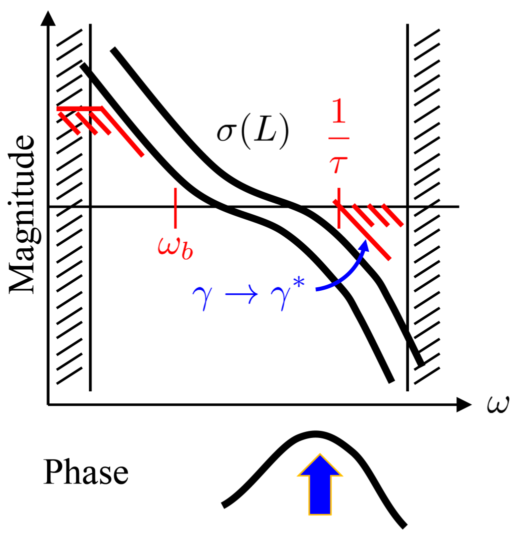


Important Note in MIMO Loop Shaping

SISO Loop Shaping



MIMO Loop Shaping



Infeasible Performance Weight

Find K s.t. $\|F_l(G,K)\|_\infty < \gamma < 1$

a) $\omega_b = 20 \quad w_p(s) = \dfrac{0.125s + 20}{s + 0.2}$

$\rightarrow \gamma = 0.6719 < 1$ ✔

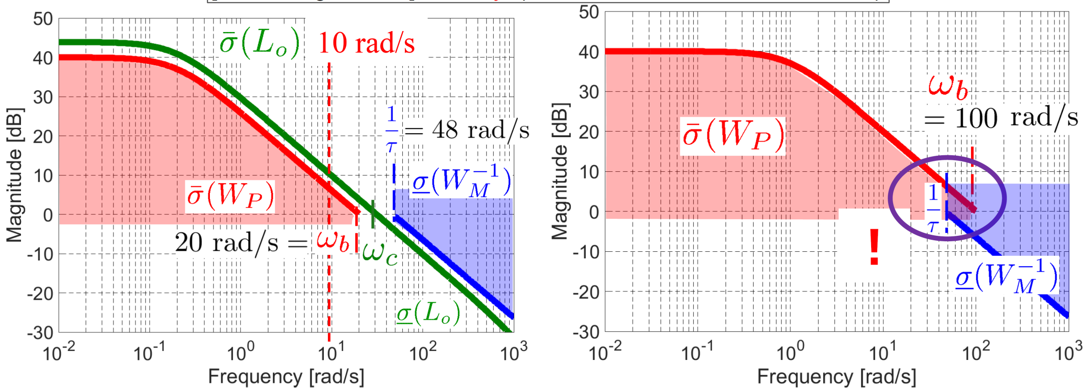
Controller exists

b) $\omega_b = 100 \quad w_p(s) = \dfrac{0.125s + 100}{s + 1}$

$\rightarrow \gamma = 1.4402 \geq 1$ ✘

No Controller

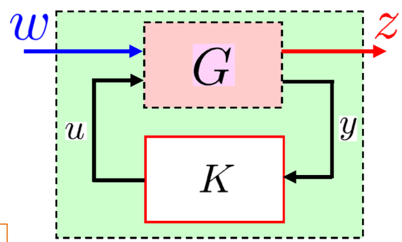
MATLAB Command
[Khi,CLhi,ghi,hiinfo] = hinfsyn(G,nmeas,ncon,'Gmax',1,'Gmin',1);



Important Note: Ensuring Assumptions

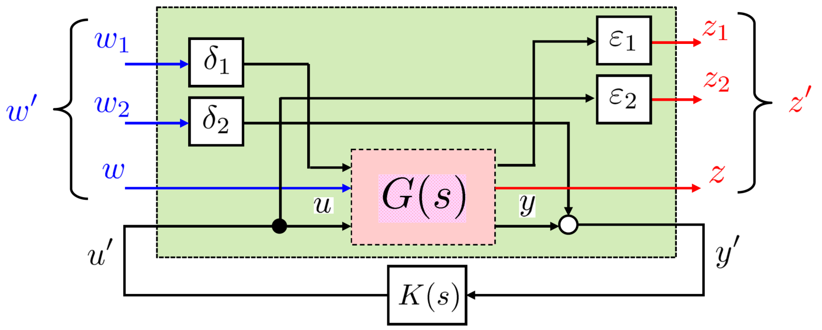
- **Generalized Plant**

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \qquad \begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w + D_{22} u \end{aligned}$$



(A1) (A, B_2) is stabilizable and (C_2, A) is detectable
(A2) (A, B_1) is controllable and (C_1, A) is observable

- Example



Example: H2 Controller

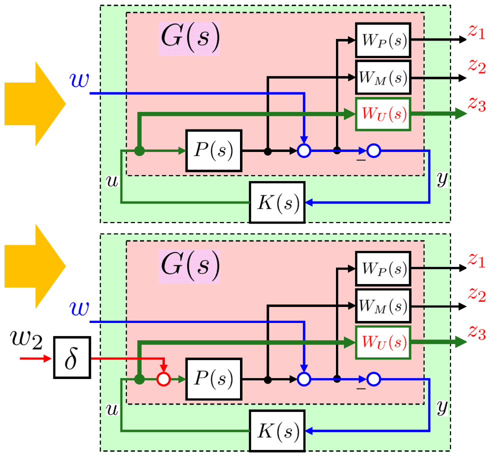
h2syn [K, CL, gam, info] = h2syn(G, nmeas, ncon)

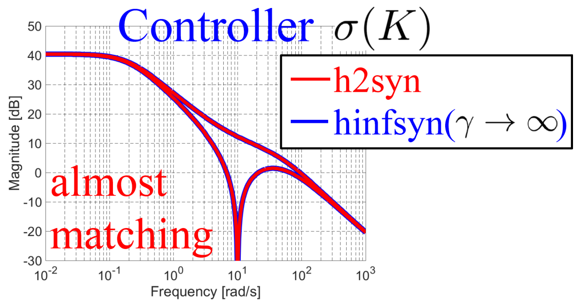
Remark:

(1) (A, B_2) :stabilizable, (C_2, A) :detectable

(2) D_{12} : full column rank, D_{21} : full row rank ➡ Error!

$u \to z$ $w \to y$





OK $\|F_l(G,K)\|_2 = \gamma = \infty$

Set $\delta = W_u = 0.0001 I_2$

Project: Report 7

Consider your dynamic system :

Using uncertainty and performance weighing functions (W2 and W1) in the previous project steps, Design an H∞ controller.

Deadline: The day before next Meeting

Please only use this email address: bevranih18@gmail.com

Thank You!