University of Kurdistan
Dept. of Control and Power Eng.

Pre-Task: 11 | **Robust Control Systems** | **H. Bevrani**

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Important Points:

1. You are free to response in Persian, English, or Kurdish (for international students)
2. Write by your own understanding (no Copy!)
3. Keep limitation words.
4. Send on the correct time, otherwise it will not be checked.
5. You can talk to each other, but you cannot send same reports.
6. Rewriting with chat GPT of a single report is easy to find.

If you have any question based on your pre-task, please write it and try to find the answer by the main class teaching, if you could not find the proper answer, please ask the lecturer in the question time of the session.

1. Based on the following selected text, provide an explanation of your understanding on the Structured Singular Value within a 350-word limit. Feel free to refer to additional references, if necessary.

21.3 The Structured Singular Value

For an unstructured perturbation, the supremum of the maximum singular value of M (*i.e.* $\|M\|_\infty$) provides a clean and numerically tractable method for evaluating robust stability. Recall that, for the standard M-Δ loop, the system fails to be robustly stable if there exists an admissible Δ such that $(I - M\Delta)$ is singular. What distinguishes the current situation from the unstructured case is that we have placed constraints on the set $\boldsymbol{\Delta}$. Given this more limited set of admissible perturbations, we desire a measure of robust stability similar to $\|M\|_\infty$. This can be derived from the *structured singular value* $\mu(M)$.

Definition 21.1 The structured singular value of a complex matrix M with respect to a class of perturbations $\boldsymbol{\Delta}$ is given by

$$\mu(M) \triangleq \frac{1}{\inf\{\sigma_{max}(\Delta) \mid \det(I - M\Delta) = 0\}}, \qquad \Delta \in \boldsymbol{\Delta}. \tag{21.4}$$

If $det(I - M\Delta) \neq 0$ for all $\Delta \in \boldsymbol{\Delta}$, then $\mu(M) = 0$.

Theorem 21.1 The M-Δ System is stable for all $\Delta \in \boldsymbol{\Delta}$ with $\|\Delta\|_\infty < 1$ if and only if

$$\sup_\omega \mu(M(j\omega)) \leq 1.$$

Proof: Immediate, from the definition. Clearly, if $\mu \leq 1$, then the norm of the smallest allowable destabilizing perturbation Δ must by definition be greater than 1.

21.4 Properties of the Structured Singular Value

It is important to note that μ is a function that depends on the perturbation class $\boldsymbol{\Delta}$ (sometimes, this function is denoted by $\mu_{\boldsymbol{\Delta}}$ to indicate this dependence). The following are useful properties of such a fucntion.

1. $\mu(M) \geq 0$.
2. If $\boldsymbol{\Delta} = \{\lambda I \mid \lambda \in \mathbb{C}\}$, then $\mu(M) = \rho(M)$, the spectral radius of M (which is equal to the magnitude of the eigenvalue of M with maximum magnitude).
3. If $\boldsymbol{\Delta} = \{\Delta \mid \Delta$ is an arbitrary complex matrix$\}$ then $\mu = \sigma_{\max}(M)$, from which $\sup_\omega \mu = \|M\|_\infty$.

Property 2 shows that the spectral radius function is a particular μ function with respect to a perturbation class consisting of matrices of the form of scaled identity. Property 3 shows that the maximum singular value function is a particular μ function with respect to a perturbation class consisting of arbitrary norm bounded perturbations (no structural constraints).

4. If $\boldsymbol{\Delta} = \{\text{diag}(\Delta_1, \ldots, \Delta_n) \mid \Delta_i \text{ complex}\}$, then $\mu(M) = \mu(D^{-1}MD)$ for any $D = \text{diag}(d_1, \ldots, d_n)$, $|d_i| > 0$. The set of such scales is denoted $\mathcal{D}$.

This can be seen by noting that $\det(I - AB) = \det(I - BA)$, so that $\det(I - D^{-1}MD\Delta) = \det(I - MD\Delta D^{-1}) = \det(I - M\Delta)$. The last equality arises since the diagonal matrices Δ and D commute.

5. If $\mathbf{\Delta} = \text{diag}(\Delta_1, \ldots, \Delta_n)$, Δ_i complex, then $\rho(M) \leq \mu(M) \leq \sigma_{\max}(M)$.

This property follows from the following observation: If $\mathbf{\Delta}_1 \subset \mathbf{\Delta}_2$, then $\mu_1 \leq \mu_2$. It is clear that the class of perturbations consisting of scaled identity matrices is a subset of $\mathbf{\Delta}$ which is a subset of the class of all unstructured perturbations.

6. From 4 and 5 we have that $\mu(M) = \mu(D^{-1}MD) \leq \inf_{D \in \mathcal{D}} \sigma_{\max}(D^{-1}MD)$.

21.5 Computation of μ

In general, there is no closed-form method for computing μ. Upper and lower bounds may be computed and refined, however. In these notes we will only be concerned with computing the upper bound. If $\mathbf{\Delta} = \text{diag}(\Delta_1, \ldots, \Delta_n)$, then the upper bound on μ is something that is easy to calculate. Furthermore, property 6 above suggests that by infimizing $\sigma_{\max}(D^{-1}MD)$ over all possible diagonal scaling matrices, we obtain a better approximation of μ. This turns out to be a convex optimization problem at each frequency, so that by infimizing over $\mathcal{D}$ at each frequency, the tightest upper bound over the set of $\mathcal{D}$ may be found for μ.

We may then ask when (if ever) this bound is tight. In other words, when is it truly a least upper bound. The answer is that for three or fewer Δ's, the bound is tight. The proof of this is involved, and is beyond the scope of this class. Unfortunately, for four or more perturbations, the bound is not tight, and there is no known method for computing μ exactly for more than three perturbations.

Ref.: M. Dahleh, et. al., Lectures on Dynamic Systems and Control, MIT, USA.

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