Intelligent Control

Takagi Sugeno Kang Fuzzy modeling

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TSK Fuzzy Systems

The ***Takagi-Sugeno-Kang fuzzy system*** is constructed from the following fuzzy IF-THEN rules:

Fuzzy sets

$$Ru^{(l)}: \; IF \;\; x_1 \;\; is \;\; A_1^l \;\; and \; .... \; and \;\; x_n \;\; is \;\; A_n^l \;\; THEN \;\; y^l = h^l\left(x_1, x_2, ..., x_n\right)$$

$$x_1, \cdots, x_n \in U = U_1 \times \cdots \times U_n$$

$$y \in V$$

Note that this form of fuzzy IF-THEN rule is known as Sugeno fuzzy rules.

TSK Fuzzy Systems

The final output of the Takagi-Sugeno fuzzy system can be inferred as the weighted average of the:

$$\boxed{f(x) = \frac{\sum_{l=1}^{N} y^l w^l}{\sum_{r=1}^{N} w^l}}$$

in which the weights are calculated as follows:

$$w^l = \prod_{i=1}^{n} \mu_{A_i^l}(x_i)$$

TSK Fuzzy Systems

This kind of fuzzy systems can be employed to represent nonlinear plant by local linear model. Consider a continuous nonlinear system of the following state space form:

$$\begin{cases} \dot{x}(t) = f\left(x(t), u(t)\right) \\ y(t) = h\left(x(t), u(t)\right) \end{cases}$$

The local linear model of fuzzy IF-THEN rules can be written as follows:

$$Ru^{(l)}: \quad IF \;\; z_1 \;\; is \;\; F_1^l \;\; and \;\; .... \;\; and \;\; z_k \;\; is \;\; F_k^l \;\; THEN \begin{cases} \dot{x}(t) = A_l x(t) + B_l u(t) \\ y(t) = C_l x(t) + D_l u(t) \end{cases}$$

Where $Z_1, \ldots, Z_k$ denote known premise variables that may be functions of the state variables, and $F_1^l, \ldots, F_k^l$ are fuzzy sets characterized by fuzzy membership functions.

TSK Fuzzy Systems

The final outputs of the fuzzy systems are inferred as follows:

$$\begin{cases} \dot{x}(t) = \sum_{l=1}^{N} \mu_l(z)\left(A_l x(t) + B_l u(t)\right) \\ y(t) = \sum_{l=1}^{N} \mu_l(z)\left(C_l x(t) + D_l u(t)\right) \end{cases}$$

Where N is the number of rule, $z = [z_1, \dots, z_k]^T$ is the vector of fuzzy inputs and $\mu_l(z)$ is a fuzzy basis function:

$$\mu_l(z) = \frac{\prod_{i=1}^{k} F_i^l(z_i)}{\sum_{l=1}^{N} \prod_{i=1}^{k} F_i^l(z_i)}$$

TSK Fuzzy Systems

Example: Consider the following nonlinear system:

$$\begin{cases} \dot{x}_1 = -x_1 + x_1 x_2^3 \\ \dot{x}_2 = -x_2 + (3 + x_2)\, x_1^3 \end{cases}$$

To construct a fuzzy local linear model, It is assumed that $x_1, x_2 \in [-1, +1]$. This nonlinear model can be easily rewritten as follows:

Premise Variables

$$\dot{x} = \begin{bmatrix} -1 & \overbrace{x_1 x_2^2}^{z_1} \\ \underbrace{(3 + x_2)\, x_1^2}_{z_2} & -1 \end{bmatrix} x$$

TSK Fuzzy Systems

----: To obtain the local linear model of the considered nonlinear system, the following fuzzy IF-THEN rules can be employed:

$$Ru^{(1)}: \quad IF\ \ z_1\ is\ F_1\ \ and\ \ z_2\ is\ G_1\ \ THEN\ \dot{x}(t) = A_1 x(t)$$

$$Ru^{(2)}: \quad IF\ \ z_1\ is\ F_1\ \ and\ \ z_2\ is\ G_2\ \ THEN\ \dot{x}(t) = A_2 x(t)$$

$$Ru^{(3)}: \quad IF\ \ z_1\ is\ F_2\ \ and\ \ z_2\ is\ G_1\ \ THEN\ \dot{x}(t) = A_3 x(t)$$

$$Ru^{(4)}: \quad IF\ \ z_1\ is\ F_2\ \ and\ \ z_2\ is\ G_2\ \ THEN\ \dot{x}(t) = A_4 x(t)$$

TSK Fuzzy Systems

----: Now, calculate the minimum and maximum of the chosen premise variables:

$$\left.\begin{array}{l} x_1 \in [-1,+1] \\ x_2 \in [-1,+1] \end{array}\right\} \Rightarrow$$

$$\min\left\{z_1 = x_1 x_2^2\right\} = -1 \qquad \max\left\{z_1 = x_1 x_2^2\right\} = +1$$

$$\min\left\{z_2 = (3+x_2)\,x_1^2\right\} = 0 \qquad \max\left\{z_2 = (3+x_2)\,x_1^2\right\} = 4$$

TSK Fuzzy Systems

Accordingly, the matrices A_1, A_2, A_3 and A_4 can be given as:

$$\left.\begin{aligned}
&\dot{x} = \begin{bmatrix} -1 & z_1 \\ z_2 & -1 \end{bmatrix} x \\
&\min\{z_1\} = -1 \\
&\max\{z_1\} = +1 \\
&\min\{z_2\} = 0 \\
&\max\{z_2\} = +4
\end{aligned}\right\}$$

$$\boxed{\begin{array}{cc} A_1 = \begin{bmatrix} -1 & 1 \\ 4 & -1 \end{bmatrix} & A_2 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \\ A_3 = \begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix} & A_4 = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \end{array}}$$

TSK Fuzzy Systems

-----: Consequently, the grades of membership of the premise variables are obtained as follows:

$$\left.\begin{array}{l}\min\{z_1\}=-1\\ \max\{z_1\}=+1\end{array}\right\} \Rightarrow \boxed{z_1=(+1)F_1(z_1)+(-1)F_2(z_1)}$$

$$\left.\begin{array}{l}\min\{z_2\}=0\\ \max\{z_2\}=+4\end{array}\right\} \Rightarrow \boxed{z_2=(+4)G_1(z_2)+(0)G_2(z_2)}$$

where

$$F_1(z_1)+F_2(z_1)=1$$

$$G_1(z_2)+G_2(z_2)=1$$

TSK Fuzzy Systems

-----: Then, by solving the obtained equations, the membership functions are as follows:

$$F_1(z_1) = \frac{1+z_1}{2} \qquad F_2(z_1) = \frac{1-z_1}{2}$$

$$G_1(z_2) = \frac{z_2}{4} \qquad G_2(z_2) = \frac{4-z_2}{4}$$

TSK Fuzzy Systems

-----: It can be checked that the designed fuzzy systems exactly represents the nonlinear system in the interval $[-1, +1] \times [-1, +1]$:

$$\boxed{\dot{x}(t) = \sum_{l=1}^{4} \mu_l(z) A_l x(t)}$$

with

$$\mu_1(z_1, z_2) = F_1(z_1) G_1(z_2)$$
$$\mu_2(z_1, z_2) = F_1(z_1) G_2(z_2)$$
$$\mu_3(z_1, z_2) = F_2(z_1) G_1(z_2)$$
$$\mu_4(z_1, z_2) = F_2(z_1) G_2(z_2)$$

TSK Fuzzy Systems

Assignment: Consider the following nonlinear system:

$$\begin{cases} \dot{x}_1 = -x_1 + x_1 x_2^3 \\ \dot{x}_1 = -x_2 + (3 + x_2)\, x_1^3 \end{cases}$$

assume that $x_1, x_2 \in [-\alpha, \alpha]$.

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- Takagi-Sugeno-Kang Fuzzy systems
- Parallel Distributed Compensator
- Design Example

Parallel distributed Compensator

To stabilize systems represented by T-S fuzzy systems of the following form:

$$Ru^{(l)}: \quad IF\ z_1\ is\ F_1^l\ and\ ....\ and\ z_k\ is\ F_k^l\ THEN\ \dot{x}(t) = A_l x(t) + B_l u(t)$$

the parallel distributed compensator controller is defined as:

$$Ru^{(m)}: \quad IF\ z_1\ is\ F_1^m\ and\ ....\ and\ z_k\ is\ F_k^m\ THEN\ u(t) = K_m x(t)$$

Parallel distributed Compensator

By applying the parallel distributed compensator control law to local linear model, the closed loop system is obtained by:

$$\boxed{\dot{x}(t)=\sum_{m=1}^{N}\sum_{l=1}^{N}\mu_m(z)\mu_l(z)\left(A_l+B_lK_m\right)x(t)}$$

which can be represented as follows:

$$\boxed{\dot{x}(t)=\sum_{l=1}^{N}\left(\mu_l(z)\right)^2G_{ll}x(t)+\sum_{l=1}^{N}\sum_{l<m}^{N}\mu_m(z)\mu_l(z)\left(G_{lm}+G_{ml}\right)x(t)}$$

with

$$G_{lm}=A_l+B_lK_m$$

Parallel distributed Compensator

Theorem: The equilibrium of closed loop fuzzy system is asymptotically stable if there exist a symmetric matrix *P>0* such that:

$$G_{ll}^T P + P G_{ll} < 0$$

$$\left(\frac{G_{lm}+G_{ml}}{2}\right)^T P + P\left(\frac{G_{lm}+G_{ml}}{2}\right) \leq 0 \;\;, l < m$$

For $\mu_m\left(z(t)\right)\mu_l\left(z(t)\right) \neq 0, \forall t$ and $l, m = 1, 2, \ldots, N$.

Parallel distributed Compensator

According to the theorem, the following linear matrix inequalities (LMIs) should be solved to design parallel distributed compensator:

$$-XA_l^T - A_l X + M_l^T B_l^T + B_l M_l > 0$$

$$-XA_l^T - A_l X - XA_m^T - A_m X + M_m^T B_l^T + B_l M_m + M_l^T B_m^T + B_m M_l \geq 0$$

$$X > 0$$

Then the controller gains is obtained:

$$P = X^{-1} \quad and \quad K_l = M_l X^{-1}$$

Content

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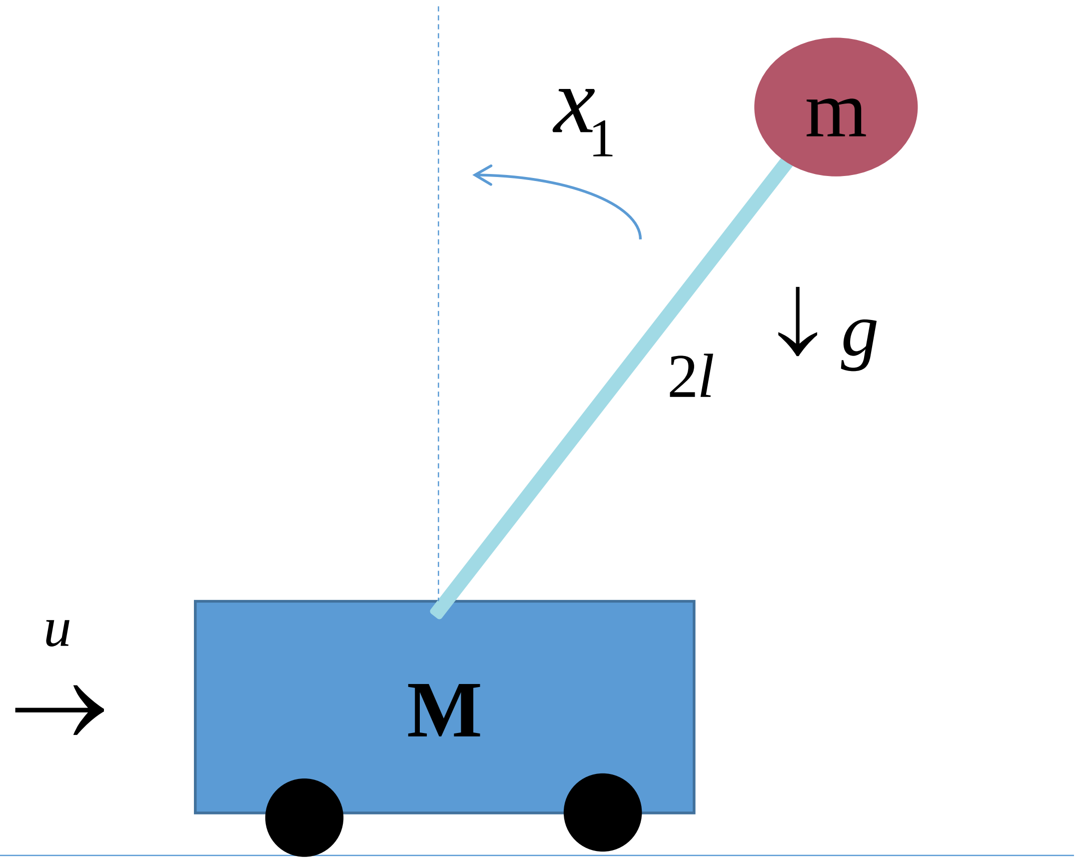
Inverted Pendulum System

$$\begin{cases} g = 9.8 \, {}^{m}\!/\!{}_{s^2} \\ m = 2\,kg \\ M = 8\,kg \\ 2l = 1m \end{cases}$$



Inverted Pendulum System

The dynamic equations of the nonlinear inverted pendulum system are as follows:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = \dfrac{g\sin\left(x_1(t)\right) - \dfrac{amlx_2^2(t)\sin\left(2x_1(t)\right)}{2}}{\dfrac{4l}{3} - aml\cos^2\left(x_1(t)\right)} + \dfrac{-a\cos\left(x_1(t)\right)}{\dfrac{4l}{3} - aml\cos^2\left(x_1(t)\right)}.u(t) \end{cases}$$

where $a = {}^{1}/_{(M+m)}$; $x_1(t)$ and $x_2(t)$ are respectively the angle and the angular velocity of the pendulum.

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Inverted Pendulum System

T-S Fuzzy Model

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Solve LMIs

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Simulate System

Inverted Pendulum System

It can be shown that this nonlinear system can be approximated by the following two rules:

$$Ru^{(1)}: \quad IF\ \ x_1\ \ is\ about\ \ 0 \quad THEN\ \dot{x}(t) = A_1 x(t) + B_1 u(t)$$

$$Ru^{(2)}: \quad IF\ \ x_1\ \ is\ about\ {}^{\pm\pi}\!/_{2}\ \ THEN\ \dot{x}(t) = A_2 x(t) + B_2 u(t)$$

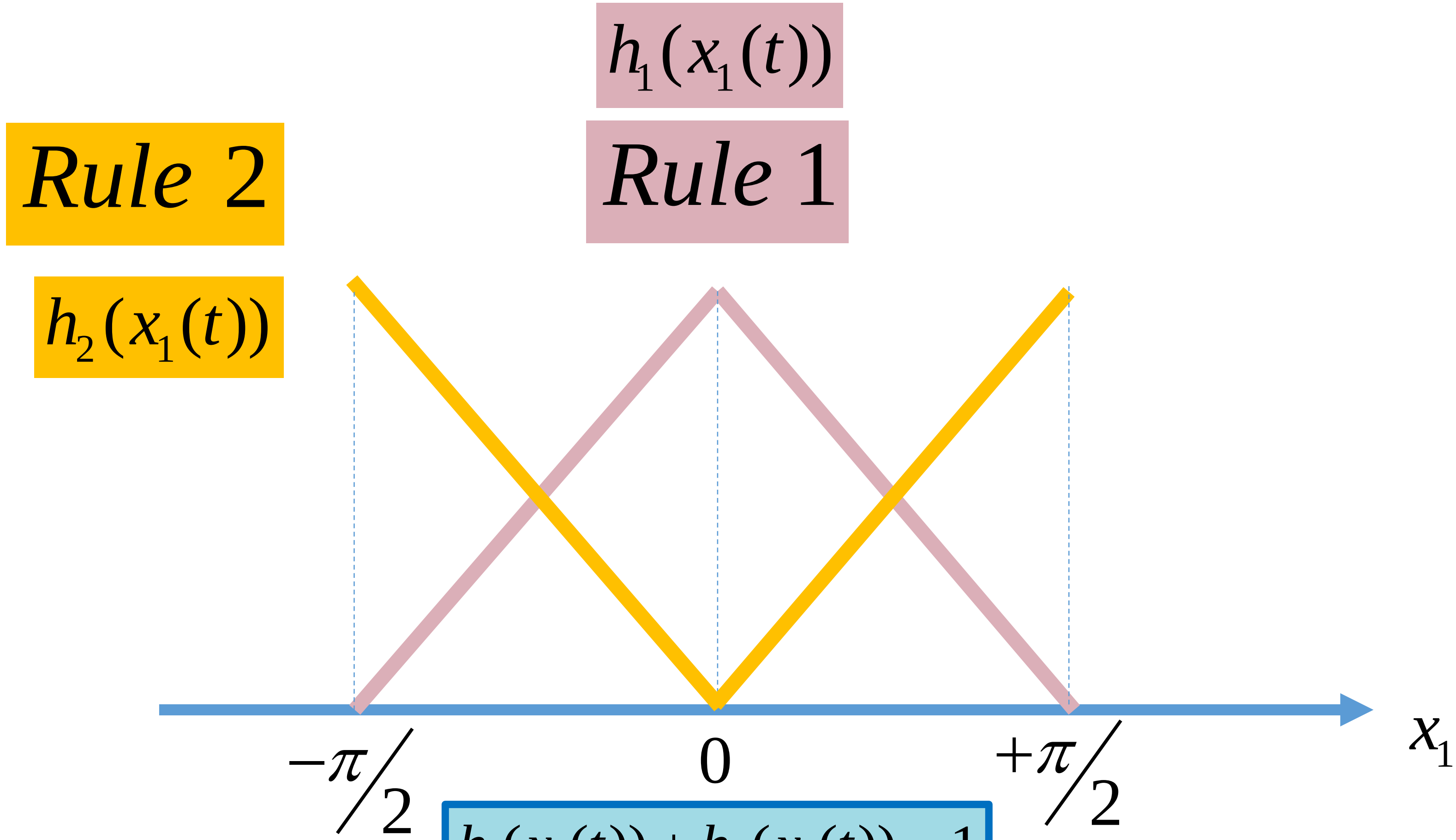
with

$$A_1 = \begin{bmatrix} 0 & 1 \\ \dfrac{g}{\dfrac{4l}{3} - aml} & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 \\ \dfrac{2g}{\pi\left(\dfrac{4l}{3} - aml\beta^2\right)} & 0 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ -\dfrac{a}{\dfrac{4l}{3} - aml} \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ -\dfrac{a\beta}{\dfrac{4l}{3} - aml\beta^2} \end{bmatrix}$$

$$\beta = \cos(88^\circ)$$

Inverted Pendulum System

Membership functions are chosen as:



$$h_1(x_1(t)) + h_2(x_1(t)) = 1$$

Inverted Pendulum System

Accordingly, by solving the following linear matrix inequality:

$$-XA_1^T - A_1X + M_1^TB_1^T + B_1M_1 > 0$$

$$-XA_2^T - A_2X + M_2^TB_2^T + B_2M_2 > 0$$

$$-XA_1^T - A_1X - XA_2^T - A_2X + M_2^TB_1^T + B_1M_2 + M_1^TB_2^T + B_2M_1 \geq 0$$

$$X = X^T > 0$$

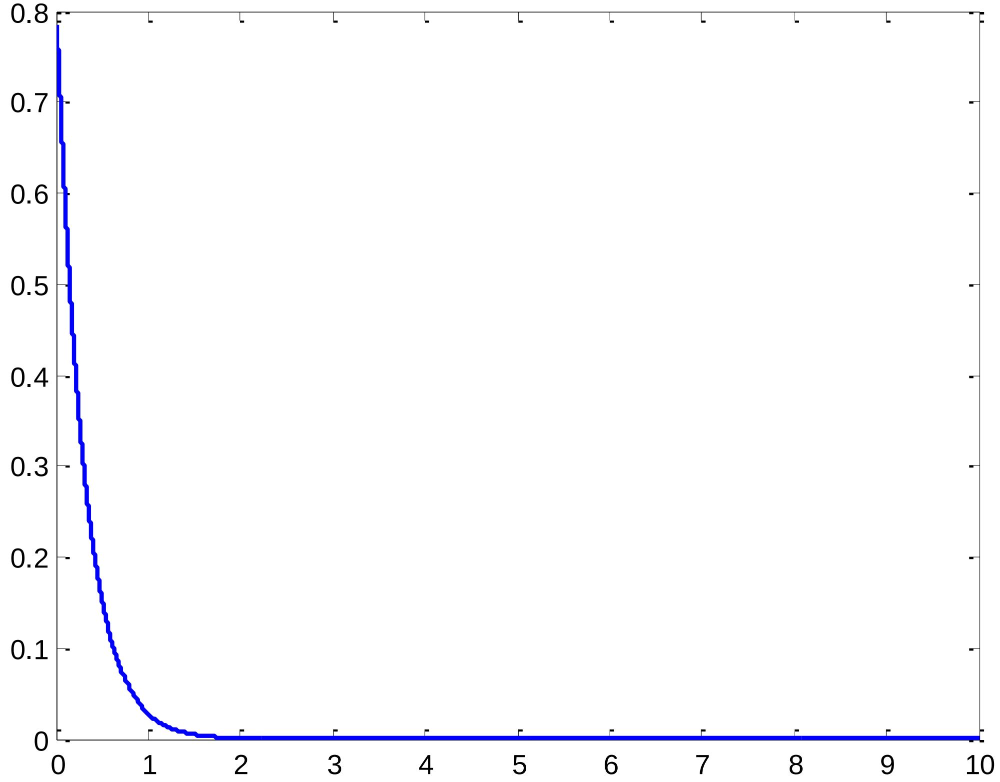
the controller gains is given by:

LMI Toolbox

$$K_1 = [\quad\quad] \quad and \quad K_2 = [\quad\quad]$$

Inverted Pendulum System

Finally, simulation result is as foolows:



Thanks