Linear Control System

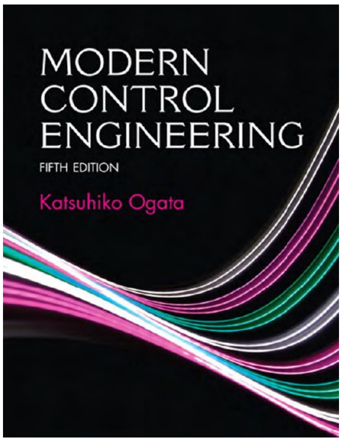
Mathematical Modeling of Dynamic Systems

Hassan Bevrani

Professor, University of Kurdistan

Spring 2024

Contents and Reference



Pages: 21-42, 81-85

Automatic Controllers

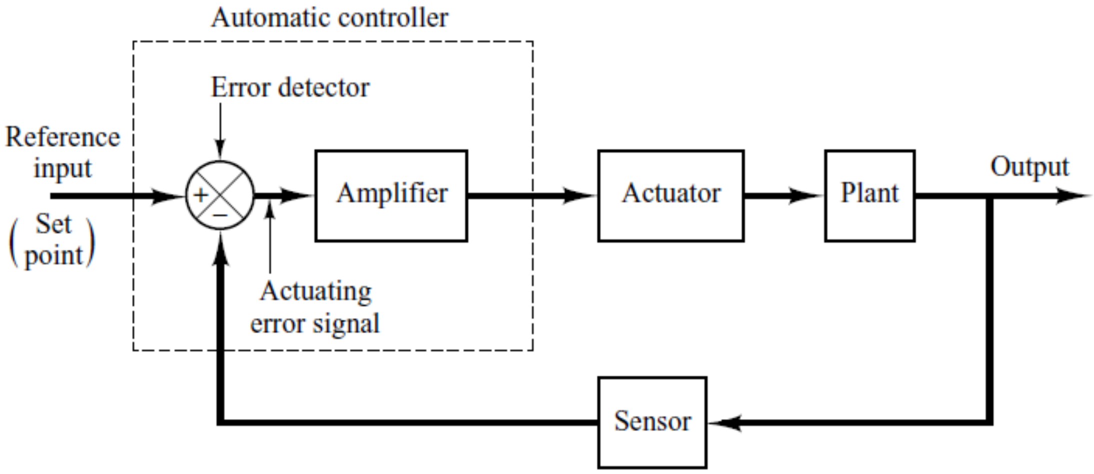
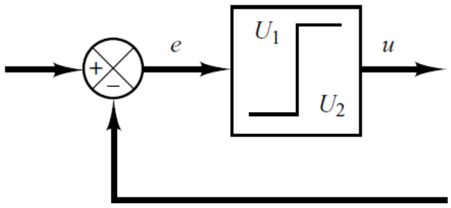


Figure 2-6 Block diagram of an industrial control system

Classifications of Industrial Controllers

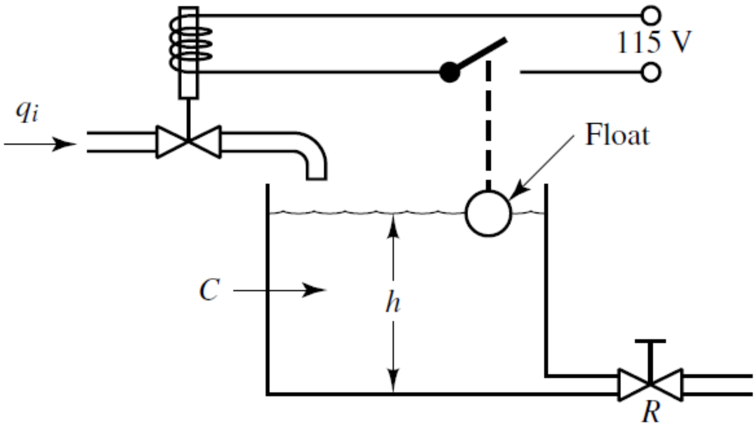
1. Two-position (on–off) controllers
2. Proportional (P) controllers
3. Integral (I) controllers
4. Proportional-integral (PI) controllers
5. Proportional-derivative (PD) controllers
6. Proportional-integral-derivative (PID) controllers

Two-Position (On-Off) Control Action



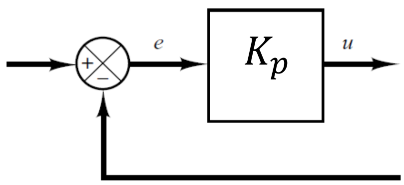
$u(t) = U_1$ for $e(t) > 0$

$u(t) = U_2$ for $e(t) < 0$



Proportional and Integral Controllers

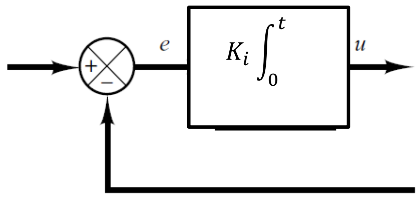
- Proportional Control Action



$u(t) = K_p e(t)$ K_p: proportional gain

$$\frac{U(s)}{E(s)} = K_p$$

- Integral Control Action



$$u(t) = K_i \int_0^t e(t)dt \qquad \frac{du(t)}{dt} = K_i e(t)$$

$$\frac{U(s)}{E(s)} = \frac{K_i}{s}$$

K_i: adjustable constant

PI and PD Controllers

- Proportional-Integral (PI) Control Action

$$u(t) = K_p e(t) + \frac{K_p}{T_i}\int_0^t e(t)dt \quad \Rightarrow \quad \frac{U(s)}{E(s)} = K_p\left(1 + \frac{1}{T_i s}\right)$$

T_i: integral time

- Proportional-Derivative (PD) Control Action

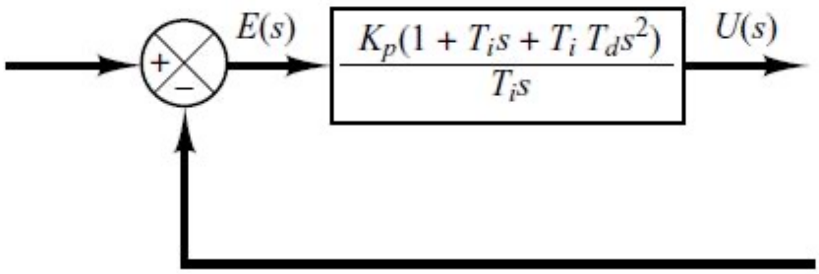
$$u(t) = K_p e(t) + K_p T_d \frac{de(t)}{dt} \quad \Rightarrow \quad \frac{U(s)}{E(s)} = K_p(1 + T_d s)$$

T_d : derivative time

PID Controllers

- Proportional-Integral-Derivative (PID) Control Action

$$u(t) = K_p e(t) + \frac{K_p}{T_i}\int_0^t e(t)dt + K_p T_d \frac{de(t)}{dt} \quad \Rightarrow \quad \frac{U(s)}{E(s)} = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)$$



Block diagram of a PID controller

Closed-Loop System Subjected to a Disturbance

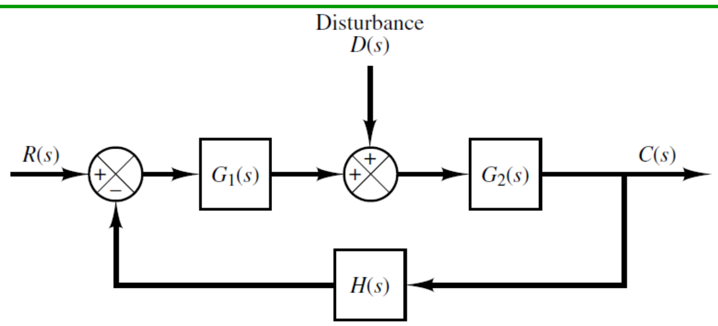


Figure 2-11

Closed-loop system subjected to a disturbance

$$\frac{C_D(s)}{D(s)} = \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)} \quad \xrightarrow{|G_1(s)G_2(s)H(s)| \gg 1} \quad \frac{C_D(s)}{D(s)} = 0$$

$$\frac{C_R(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)} \quad \xrightarrow{|G_1(s)G_2(s)H(s)| \gg 1} \quad \frac{C_R(s)}{R(s)} = \frac{1}{H(s)}$$

$$C(s) = C_R(s) + C_D(s) = \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}[G_1(s)R(s) + D(s)]$$

The closed-loop system becomes independent of G_1(s) and G_2(s) and inversely proportional to H(s), so that the variations of *system parameters* do not affect the closed-loop transfer function.

Drawing a Block Diagram

1. Write the equations describing the dynamic behavior of each component.

2. Take the Laplace transforms of these equations, assuming zero initial conditions,

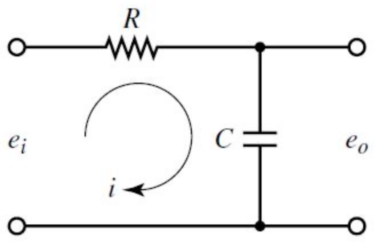
3. Represent each Laplace-transformed equation individually in block form.

4. Finally, assemble the elements into a complete block diagram.

Example:

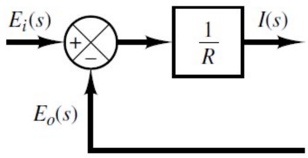
1. $i = \frac{e_i - e_o}{R}$ $\qquad e_o = \frac{\int i\,dt}{C}$

2. $I(s) = \frac{E_i(s) - E_o(s)}{R}$ $\qquad E_o(s) = \frac{I(s)}{Cs}$



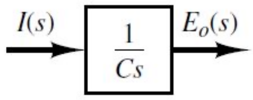
(a) RC circuit

3. $I(s) = \dfrac{E_i(s) - E_o(s)}{R}$



(b) Block diagram representing Equation (2-6)

$E_o(s) = \dfrac{I(s)}{Cs}$



(c) Block diagram representing Equation (2-7)

4.

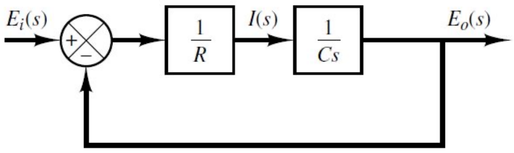
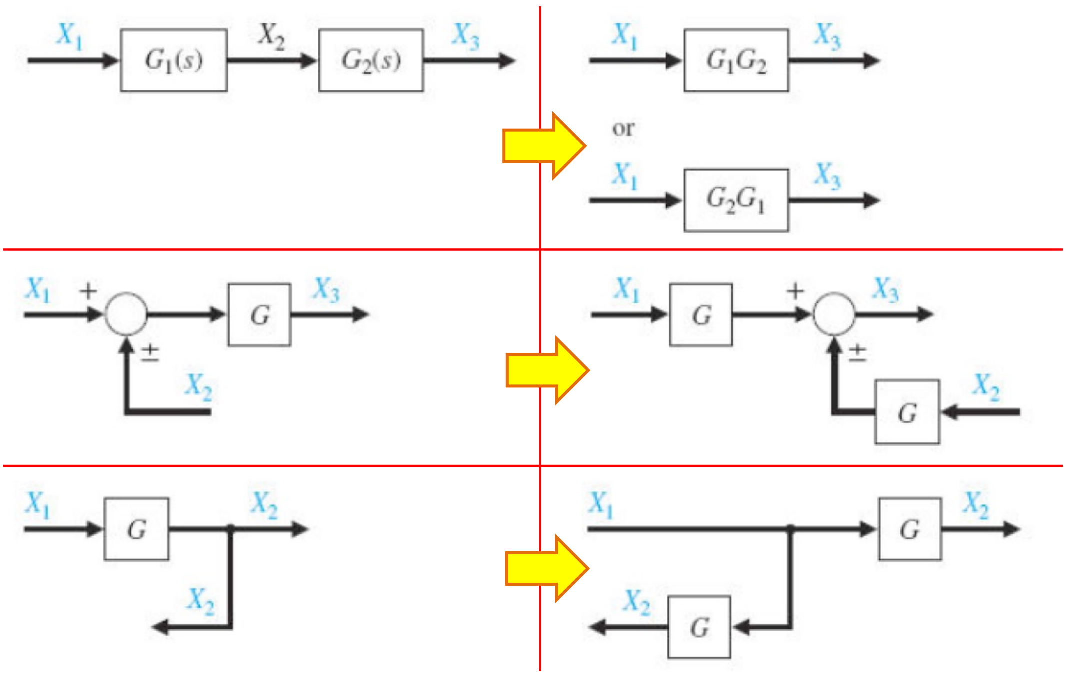


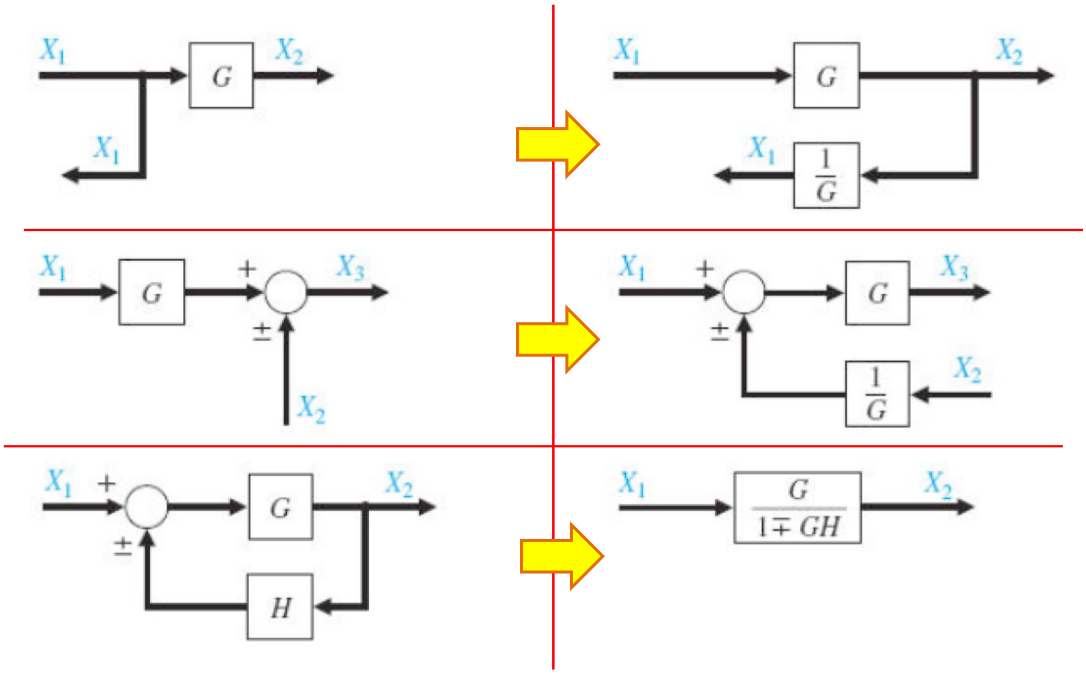
Figure 2-12 (d) Block diagram of the RC circuit

Note: In case of possibility, the block diagram can be reduced (simplified).

Block Diagram Simplification

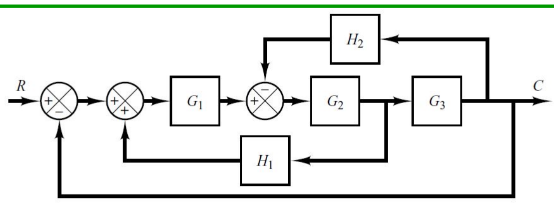


Block Diagram Simplification



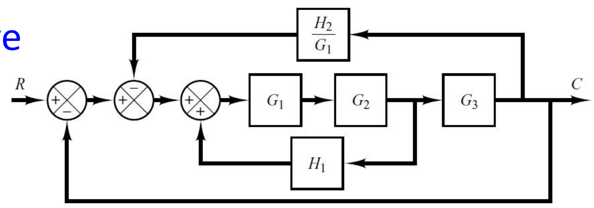
Example

Simplify this diagram:



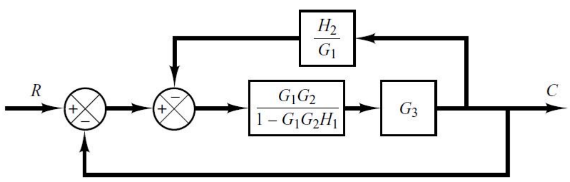
(a) Multiple-loop system

① Moving the summing point of the negative feedback loop containing *H*2 outside the positive feedback loop containing *H*1



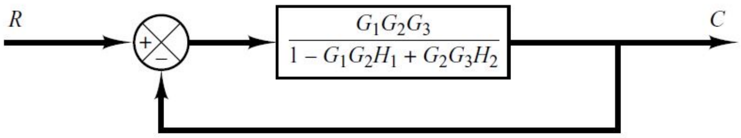
(b) Successive reductions of the block diagram

② Eliminating the positive feedback loop



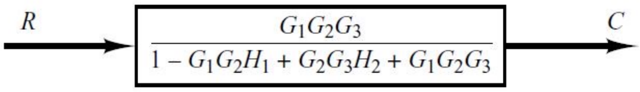
(c) Successive reductions of the block diagram

③ **Elimination of the loop containing *H2/G1***



(d) Successive reductions of the block diagram

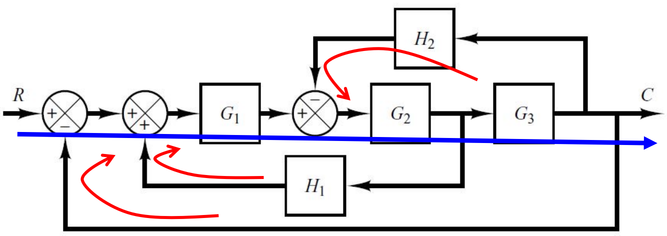
④ **Eliminating the feedback loop**

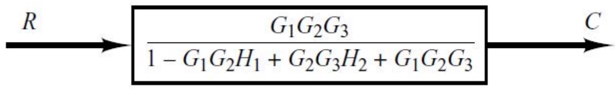


(e) Successive reductions of the block diagram

Transfer function: $\frac{C(s)}{R(s)}$ ⇨ $\frac{C(s)}{R(s)} = \frac{G_1G_2G_3}{1-G_1G_2H_1+G_2G_3H_2+G_1G_2G_3}$

Easier Solution: Mason`s Theorem





Transfer function: $\frac{C(s)}{R(s)}$

Numerator: The product of the transfer functions of the feedforward path

Denominator: $1-\sum$(product of the transfer functions around each loop)

$= 1 - G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3$

Modeling in State Space

State

The state of a dynamic system is the smallest set of variables (called *state variables*) such that knowledge of these variables at $t = t_0$, together with knowledge of the input for $t > t_0$, completely determines the behavior of the system for any time.

State Variables

The variables making up the smallest set of variables $(x_1, x_2, \ldots, x_n)$ that determine the state of the dynamic system.

State Vector

A vector that determines uniquely the system state $\boldsymbol{x}(t) = [x_1 \; x_2 \; \ldots \; x_n]$ for any time.

State Space

The n-dimensional space whose coordinate axes consist of the x_1 axis,x_2 axis, …, x_n axis. Any state can be represented by a point in the state space.

State-Space Equations

Assume a system with

r inputs $\left(u_1(t), u_2(t), \ldots, u_r(t)\right)$
m uoutputs $(y_1(t), y_2(t), \ldots, y_m(t))$

Then the system may be described by

$$\begin{aligned} \dot{x}_1(t) &= f_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ \dot{x}_2(t) &= f_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ &\vdots \\ \dot{x}_n(t) &= f_n(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \end{aligned} \quad (2\text{-}8)$$

$$\begin{aligned} y_1(t) &= g_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ y_2(t) &= g_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ &\vdots \\ y_m(t) &= g_m(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \end{aligned} \quad (2\text{-}9)$$

Continue

Then Equations (2–8) and (2–9) become

State equation $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$ (2-10)
Output equation $\mathbf{y}(t) = \mathbf{g}(\mathbf{x}, \mathbf{u}, t)$ (2-11)

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \qquad \mathbf{f}(\mathbf{x}, \mathbf{u}, t) = \begin{bmatrix} f_1(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t) \\ f_2(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t) \\ \vdots \\ f_n(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t) \end{bmatrix}$$

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_m(t) \end{bmatrix} \qquad \mathbf{g}(\mathbf{x}, \mathbf{u}, t) = \begin{bmatrix} g_1(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t) \\ g_2(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t) \\ \vdots \\ g_m(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t) \end{bmatrix} \qquad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_r(t) \end{bmatrix}$$

Continue

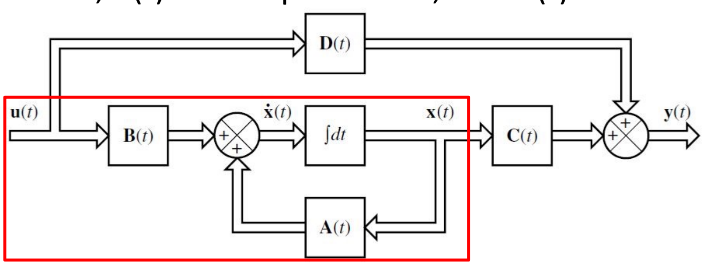
Linearized state equation and output equation:

State equation $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$ (2-12)
Output equation $\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$ (2-13)

A(t) is called the state matrix, **B**(t) the input matrix, **C**(t) the output matrix, and **D**(t) the direct transmission matrix

Figure 2-14

Block diagram of the linear , continuous-time control system represented in state space



If vector functions **f** and **g** do not involve time t explicitly then the system is called a time-invariant system.

State equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ (2-14)
Output equation $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$ (2-15)

Example

Consider the mechanical system shown in Figure:

From the diagram, the system equation is

$$m\ddot{y} + b\dot{y} + ky = u$$

This system is of second order. Let us define state variables as

$$x_1(t) = y(t)$$
$$x_2(t) = \dot{y}(t)$$

Then we obtain

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{m}(-ky - b\dot{y}) + \frac{1}{m}u$$

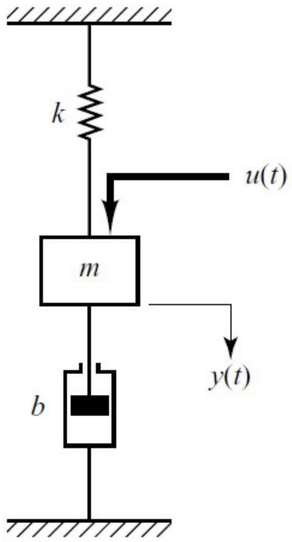


Figure 2-15

Mechanical system

or

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$

The output equation is

$$y = x_1$$

In a vector-matrix form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{b}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

in the standard form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} + Du$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D = 0$$

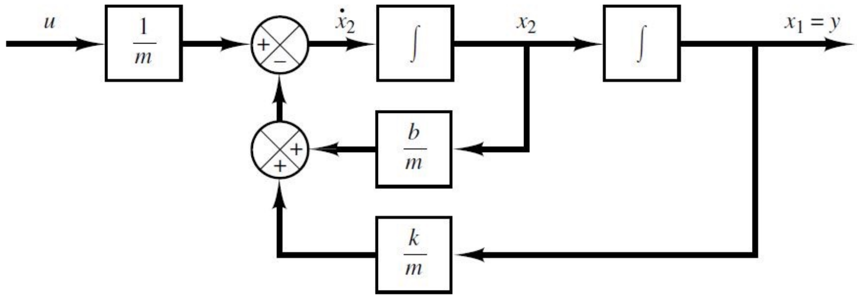


Figure 2-16 Block diagram of the mechanical system shown in Figure 2-15

Transfer Functions and State-Space Equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} + Du$$

$$\frac{Y(s)}{U(s)} = G(s)$$

The Laplace transforms of state-space equations

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s)$$
$$Y(s) = \mathbf{C}\mathbf{X}(s) + DU(s)$$

Assuming $\mathbf{x}(0)$

$$s\mathbf{X}(s) - \mathbf{A}\mathbf{X}(s) = \mathbf{B}U(s)$$
$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}U(s)$$

By pre-multiplying $(s\mathbf{I} - \mathbf{A})^{-1}$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s)$$

By substituting above equation into $Y(s) = \mathbf{C}\mathbf{X}(s) + DU(s)$

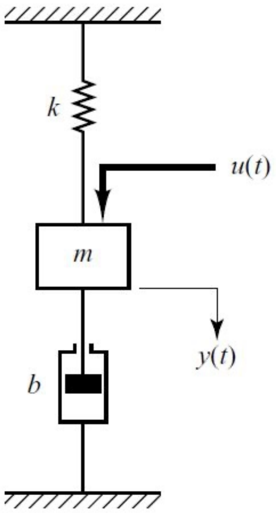
$$Y(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D]U(s)$$

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

Example

Consider again this mechanical system

By substituting **A**, **B**, **C**, and **D** into Equation (2–29)



$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

$$G(s) = [1 \quad 0]\left\{\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}\right\}^{-1}\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} + 0$$

$$= [1 \quad 0]\begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$\begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1} = \frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}}\begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix}$$

$$= [1 \quad 0]\frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}}\begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix}\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} = \frac{1}{ms^2 + bs + k}$$

Transfer Matrix

Consider a multiple-input, multiple-output system with ***r*** inputs and ***m*** outputs

$$\mathbf{u}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_m\end{bmatrix}\qquad \mathbf{y}=\begin{bmatrix}y_1\\y_2\\\vdots\\y_m\end{bmatrix}$$

The transfer matrix **G**(s) relates the output **Y**(s) to the input **U**(s), or

$$\mathbf{Y}(s)=\mathbf{G}(s)\mathbf{U}(s)$$
$$\mathbf{G}(s)=\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}$$

$\mathbf{G}(s)$ is an m × r matrix

State-Space Representation of *n*-Order Systems

- The forcing function does not involve derivative terms

Consider the following nth-order system:

$$y^{(n)}+a_1y^{(n-1)}+\cdots+a_{n-1}\dot{y}+a_ny=u \qquad \text{(2-30)}$$

Let us define $x_1=y\,,\quad x_2=\dot{y}\,,\quad \cdots,\quad x_n=y^{(n-1)}$

(2–30) can be written as $\dot{x}_1=x_2\,,\ \dot{x}_2=x_3\,,\ \cdots,\ \dot{x}_{n-1}=x_n\,,\quad \dot{x}_n=-a_nx_1-\cdots-a_1x_n+u$

or

$$\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}+\mathbf{B}u \qquad \text{(2-31)}$$

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}\quad \mathbf{A}=\begin{bmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&\vdots&&\vdots\\0&0&0&\cdots&1\\-a_n&-a_{n-1}&-a_{n-2}&\cdots&-a_1\end{bmatrix}\mathbf{B}=\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}$$

The output can be given by

$$y = \mathbf{Cx} \qquad \text{(2-32)} \qquad [D \text{ is zero}]$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$

Transfer function representation

$$\frac{Y(s)}{U(s)} = \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

State-Space Representation of *n*-Order Systems

- The forcing function involves derivative terms

Consider the system that involves derivatives of the forcing function, such as

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u \qquad \text{(2-33)}$$

The main problem in defining the state variables is the derivative terms of the input u. A solution is to define the following n variables as a set of n state variables:

$$\begin{aligned} x_1 &= y - \beta_0 u \\ x_2 &= \dot{y} - \beta_0 \dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u \\ x_3 &= \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u \\ &\vdots \\ x_n &= y^{(n-1)} - \beta_0 u^{(n-1)} - \beta_1 u^{(n-2)} - \cdots - \beta_{n-2}\dot{u} - \beta_{n-1} u = \dot{x}_{n-1} - \beta_{n-1} u \end{aligned} \qquad \text{(2-34)}$$

β parameters are determined from

$$\begin{aligned} \beta_0 &= b_0 \\ \beta_1 &= b_1 - a_1\beta_0 \\ \beta_2 &= b_2 - a_1\beta_1 - a_2\beta_0 \\ &\vdots \\ \beta_{n-1} &= b_{n-1} - a_1\beta_{n-2} - \cdots - a_{n-2}\beta_1 - a_{n-1}\beta_0 \end{aligned}$$

$$\begin{aligned}\dot{x}_1 &= x_2 + \beta_1 u \\ \dot{x}_2 &= x_3 + \beta_2 u \\ &\vdots \\ \dot{x}_{n-1} &= x_n + \beta_{n-1} u \\ \dot{x}_n &= -a_1 x_1 - a_{n-1} x_2 - \cdots - a_1 x_n + \beta_n u\end{aligned} \qquad (2\text{-}36)$$

Equation (2–36) and the output equation can be written as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \vdots\\ \dot{x}_{n-1}\\ \dot{x}_n\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_{n-1}\\ x_n\end{bmatrix} + \begin{bmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_{n-1}\\ \beta_n\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix} + \beta_0 u$$

or

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \qquad (2\text{-}37)$$

$$y = \mathbf{C}\mathbf{x} + Du \qquad (2\text{-}38)$$

$$\mathbf{x} = \begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_{n-1}\\ x_n\end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1\end{bmatrix} \quad \mathbf{B} = \begin{bmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_{n-1}\\ \beta_n\end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix} \qquad D = \beta_0 = b_0$$

matrices **A** and **C** are exactly the same as those for the system of Equation (2–30).The derivatives on the right-hand side of Equation (2–33) affect only the elements of the **B** matrix.

Modeling in MATLAB

Transformation from Transfer Function to State Space Representation

$$\frac{Y(s)}{U(s)} = \frac{\text{num}}{\text{den}} \Rightarrow [\text{A, B, C, D}] = \text{tf2ss}(\text{num, den})$$

$$\frac{Y(s)}{U(s)} = \frac{s}{(s+10)(s^2+4s+16)} = \frac{s}{s^3+14s^2+56s+160} \qquad (2\text{-}39)$$

One of possible state-space representations for this system is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-14 & -56 & -160\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}u \qquad (2\text{-}40)$$

$$y = [0 \quad 1 \quad 0]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + [0]u \qquad (2\text{-}41)$$

Continue

MATLAB Program 2–2

```
num = [1   0];
den = [1   14   56   160];
[A, B, C, D] = tf2ss(num, den)
```

$$\frac{Y(s)}{U(s)} = \frac{s}{s^3+14s^2+56s+160}$$

Transformation from State Space to Transfer Function

$$[\text{num, den}] = \text{ss2tf}(\text{A, B, C, D, iu})$$

iu must be specified for systems with more than one input. For example, if the system has three inputs (u1, u2, u3), then **iu** must be either 1, 2, or 3, where

$$u_1: \text{iu} = 1 \qquad u_2: \text{iu} = 2 \qquad u_3: \text{iu} = 3$$

If the system has only one input,

$$[\text{num, den}] = \text{ss2tf}(\text{A, B, C, D}) \quad \text{or} \quad [\text{num, den}] = \text{ss2tf}(\text{A, B, C, D}, 1)$$

Example

Obtain the transfer function of the system defined by the following state-space equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -25 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 25 \\ -120 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

MATLAB Program 2-3 will produce the transfer function for the given system

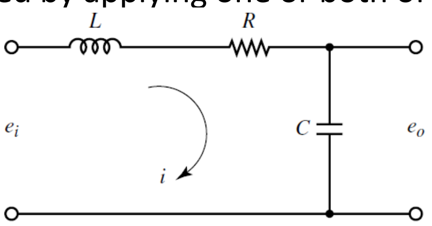
$$\frac{Y(s)}{U(s)} = \frac{25s + 5}{s^3 + 5s^2 + 25s + 5}$$

MATLAB Program 2–3

```
A = [0   1   0; 0   0   0; −5   −25   −5];
B = [0; 25; −120];
C = [1   0   0];
D = [0];
[num, den] = ss2tf(A, B, C, D)
```


Mathematical Modeling of Electrical Systems

A mathematical model of an electrical circuit can be obtained by applying one or both of Kirchhoff's laws



○ **LRC Circuit**

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = e_i$$

$$\frac{1}{C}\int i\,dt = e_o$$

⇨

$$LsI(s) + RI(s) + \frac{1}{C}\frac{1}{s}I(s) = E_i(s)$$

$$\frac{1}{C}\frac{1}{s}I(s) = E_o(s)$$

⇨

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1}$$

⇩

$$\ddot{e}_o + \frac{R}{L}\dot{e}_o + \frac{1}{LC}e_o = \frac{1}{LC}e_i$$

Assume: $x_1 = e_o$, $x_2 = \dot{e}_o$ and $u = e_i$, $y = e_o = x_1$ ⇨

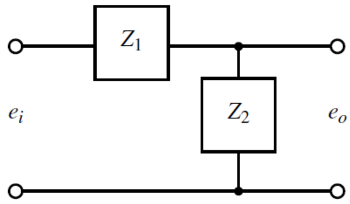
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix} u$$

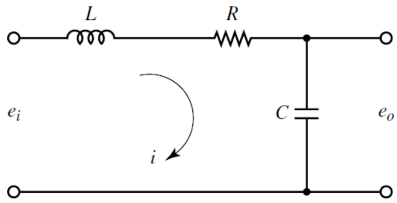
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Continue

Complex Impedances Approach

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}$$



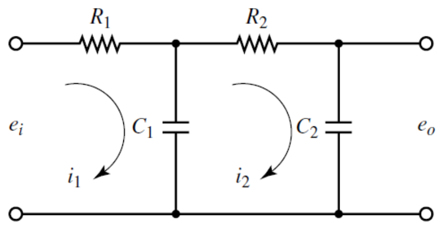


$$Z_1 = Ls + R, \qquad Z_2 = \frac{1}{Cs}$$

$$\frac{E_o(s)}{E_i(s)} = \frac{\dfrac{1}{Cs}}{Ls + R + \dfrac{1}{Cs}} = \frac{1}{LCs^2 + RCs + 1}$$

Transfer Functions of Cascaded Elements

Example



$$\frac{1}{C_1}\int (i_1 - i_2)\,dt + R_1 i_1 = e_i$$

$$\frac{1}{C_1}\int (i_2 - i_1)\,dt + R_2 i_2 + \frac{1}{C_2}\int i_2\,dt = 0$$

$$\frac{1}{C_2}\int i_2\,dt = e_o$$

$$\frac{1}{C_1 s}[I_1(s) - I_2(s)] + R_1 I_1(s) = E_i(s)$$

$$\frac{1}{C_1 s}[I_2(s) - I_1(s)] + R_2 I_2(s) + \frac{1}{C_2 s} I_2(s) = 0$$

$$\frac{1}{C_2 s} I_2(s) = E_o(s)$$

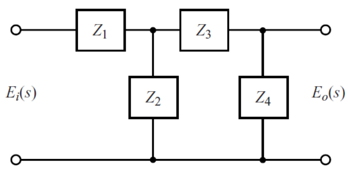
$$\frac{E_o(s)}{E_i(s)} = \frac{1}{R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_1 C_2)s + 1}$$

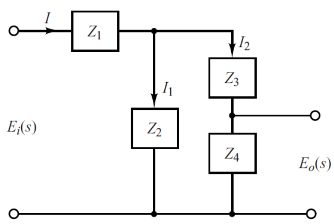
Using Complex Impedances Approach

$$Z_2 I_1 = (Z_3 + Z_4) I_2, \qquad I_1 + I_2 = I \qquad I_1 = \frac{Z_3 + Z_4}{Z_2 + Z_3 + Z_4} I, \qquad I_2 = \frac{Z_2}{Z_2 + Z_3 + Z_4} I$$

$$E_i(s) = Z_1 I + Z_2 I_1 = \left[Z_1 + \frac{Z_2(Z_3 + Z_4)}{Z_2 + Z_3 + Z_4}\right] I$$

$$E_o(s) = Z_4 I_2 = \frac{Z_2 Z_4}{Z_2 + Z_3 + Z_4} I$$





$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2 Z_4}{Z_1(Z_2 + Z_3 + Z_4) + Z_2(Z_3 + Z_4)}$$

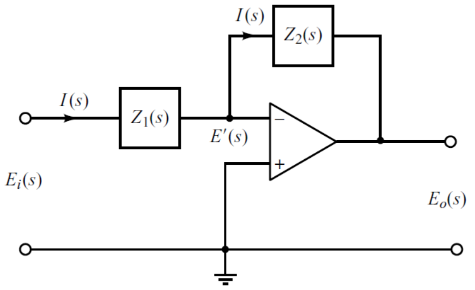
Continue

- **Op-Amp Circuits**

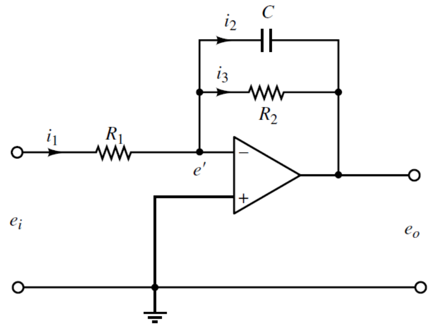
$$\frac{E_i(s)-E'(s)}{Z_1}=\frac{E'(s)-E_o(s)}{Z_2}$$

Assume E′(s)=0:

$$\frac{E_o(s)}{E_i(s)}=-\frac{Z_2(s)}{Z_1(s)}$$



$$Z_1(s)=R_1,\qquad Z_2(s)=\frac{1}{Cs+\frac{1}{R_2}}=\frac{R_2}{R_2Cs+1}$$

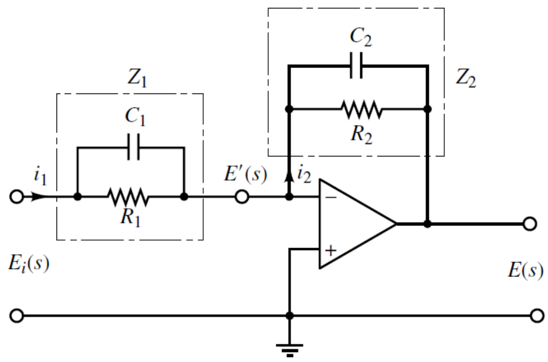


$$\Rightarrow\quad \frac{E_o(s)}{E_i(s)}=-\frac{Z_2(s)}{Z_1(s)}=-\frac{R_2}{R_1}\frac{1}{R_2Cs+1}$$

Continue

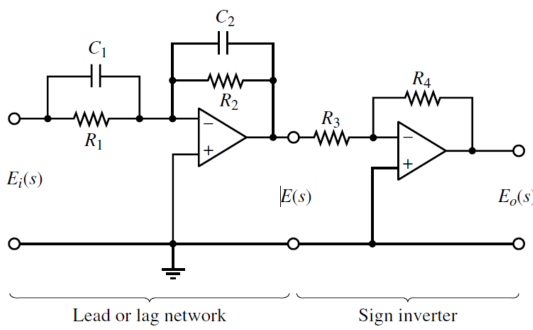
$$Z_1=\frac{R_1}{R_1C_1s+1},Z_2=\frac{R_2}{R_2C_2s+1}$$

$$\frac{E(s)}{E_i(s)}=-\frac{Z_2}{Z_1}=-\frac{R_2}{R_1}\frac{R_1C_1s+1}{R_2C_2s+1}=-\frac{C_1}{C_2}\frac{s+\frac{1}{R_1C_1}}{s+\frac{1}{R_2C_2}}$$



$$\frac{E_o(s)}{E(s)}=-\frac{R_4}{R_3}$$

$$\frac{E_o(s)}{E_i(s)}=\frac{E(s)}{E_i(s)}\frac{E_o(s)}{E(s)}=\frac{R_2R_4}{R_1R_3}\frac{R_1C_1s+1}{R_2C_2s+1}=\frac{R_4C_1}{R_3C_2}\frac{s+\frac{1}{R_1C_1}}{s+\frac{1}{R_2C_2}}$$



Assuming T=R_1C_1, αT=R_2C_2, $K_c=\frac{R_4C_1}{R_3C_2}$

$$\frac{E_o(s)}{E_i(s)}=K_c\alpha\frac{Ts+1}{\alpha Ts+1}=K_c\frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}}$$

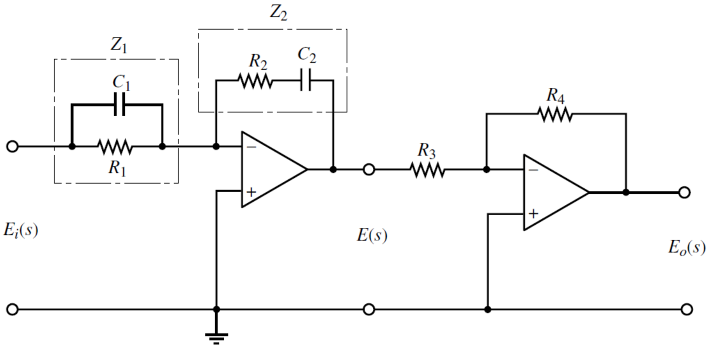
PID Controller Using Op-Amp

$$Z_1 = \frac{R_1}{R_1C_1s+1}, Z_2 = \frac{R_2C_2s+1}{C_2s}$$

$$\frac{E(s)}{E_i(s)} = -\frac{Z_2}{Z_1} = \left(\frac{R_2C_2s+1}{C_2s}\right)\left(\frac{R_1C_1s+1}{R_1}\right)$$

$$Z_3 = R_3, Z_4 = R_4$$

$$\frac{E_o(s)}{E(s)} = -\frac{Z_2}{Z_1} = -\frac{R_4}{R_3}$$



$$\frac{E_o(s)}{E_i(s)} = \frac{E(s)}{E_i(s)}\frac{E_o(s)}{E(s)} = \frac{R_4}{R_3}\frac{R_2}{R_1}\frac{(R_1C_1s+1)(R_2C_2s+1)}{R_2C_2s}$$

$$= \frac{R_4}{R_3}\frac{R_2}{R_1}\left(\frac{R_1C_1+R_2C_2}{R_2C_2} + \frac{1}{R_2C_2s} + R_1C_1s\right)$$

$$= \frac{R_4(R_1C_1+R_2C_2)}{R_3R_1C_2}\left[1 + \frac{1}{(R_1C_1+R_2C_2)s} + \frac{R_1C_1R_2C_2}{R_1C_1+R_2C_2}s\right] \quad (3\text{-}37)$$

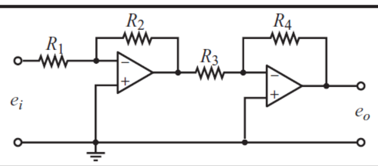
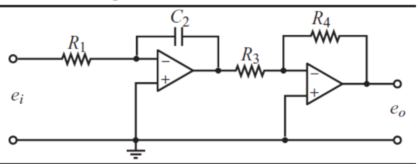
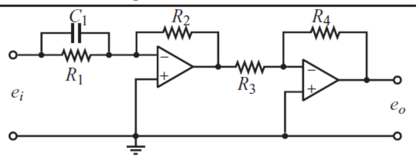
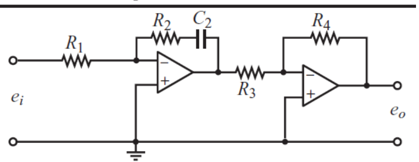
$$K_p = \frac{R_4(R_1C_1+R_2C_2)}{R_3R_1C_2}$$

$$\frac{E_o(s)}{E_i(s)} = K_p\left(1 + \frac{T_i}{s} + T_d s\right) \Rightarrow \frac{E_o(s)}{E_i(s)} = K_p + \frac{K_i}{s} + K_d s$$

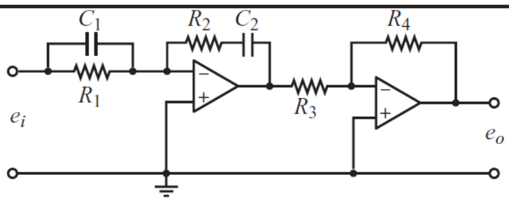
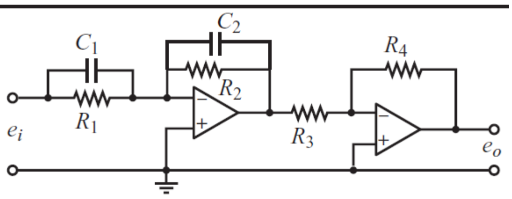
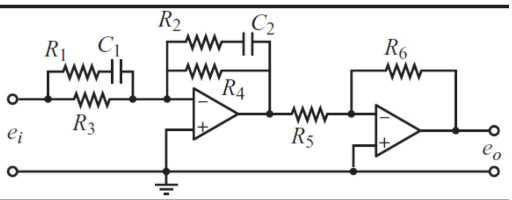
$$K_i = \frac{R_4}{R_3R_1C_2}$$

$$K_d = \frac{R_4R_2C_1}{R_3}$$

More Controllers With Op-Amp

Control Action	$G(s) = \frac{E_o(s)}{E_i(s)}$	Operational-Amplifier Circuits
P	$\frac{R_4}{R_3}\frac{R_2}{R_1}$	
I	$\frac{R_4}{R_3}\frac{1}{R_1C_2s}$	
PD	$\frac{R_4}{R_3}\frac{R_2}{R_1}(R_1C_1s+1)$	
PI	$\frac{R_4}{R_3}\frac{R_2}{R_1}\frac{R_2C_2s+1}{R_2C_2s}$	

Continue

PID	$\frac{R_4}{R_3}\frac{R_2}{R_1}\frac{(R_1C_1s+1)(R_2C_2s+1)}{R_2C_2s}$	
Lead or lag	$\frac{R_4}{R_3}\frac{R_2}{R_1}\frac{R_1C_1s+1}{R_2C_2s+1}$	
Lag–lead	$\frac{R_6}{R_5}\frac{R_4}{R_3}\frac{[(R_1+R_3)C_1s+1](R_2C_2s+1)}{(R_1C_1s+1)[(R_2+R_4)C_2s+1]}$	

Thank You!