## Linear Control System

# Transient and Steady-State Response Analyses

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MODERN CONTROL ENGINEERING

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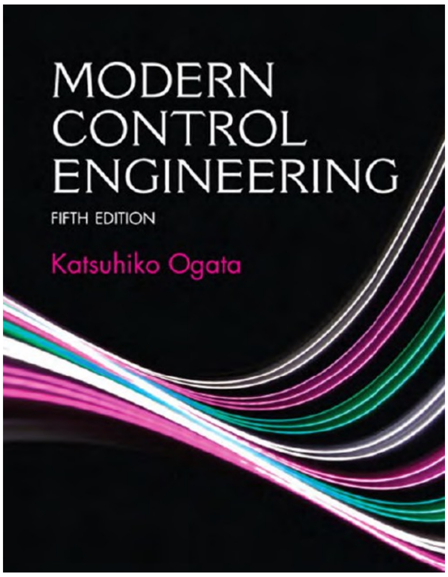
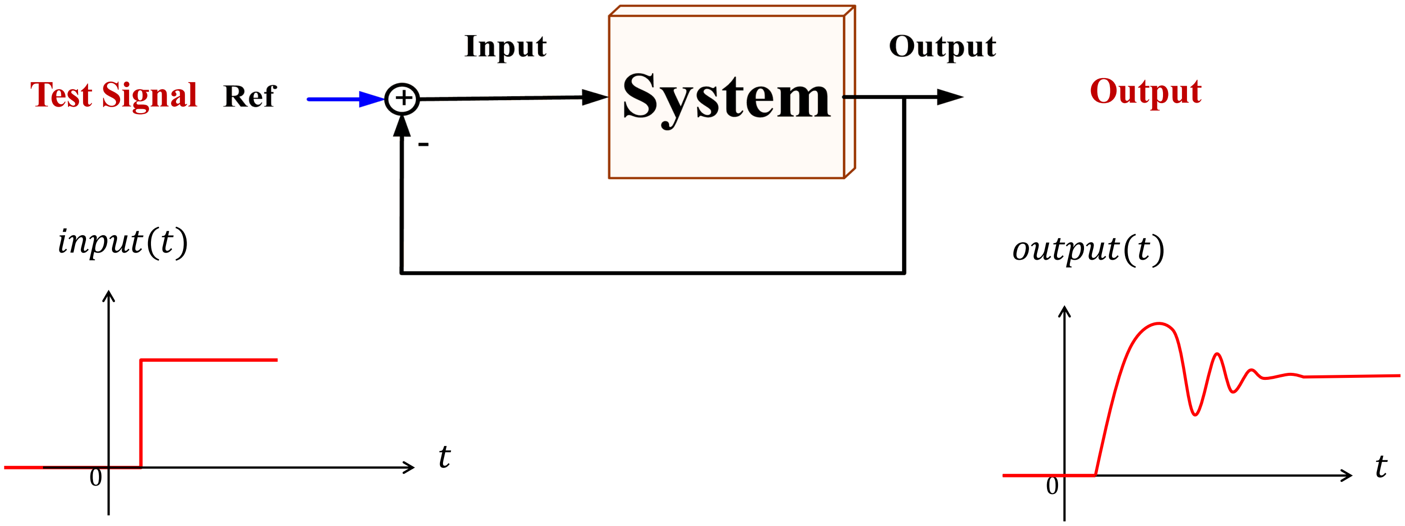


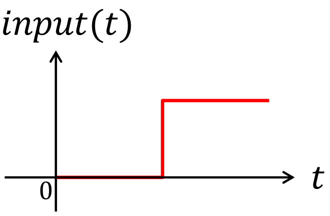
MODERN
CONTROL
ENGINEERING
FIFTH EDITION
Katsuhiko Ogata

## System Response to a Test Signal

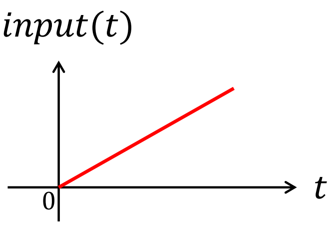


## Typical Test Signals

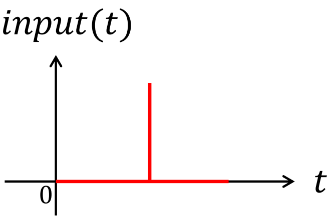
- Step



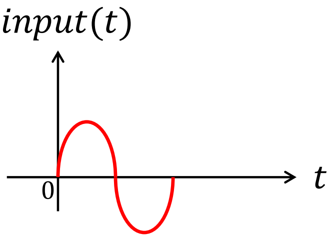
- Ramp



- Impulse

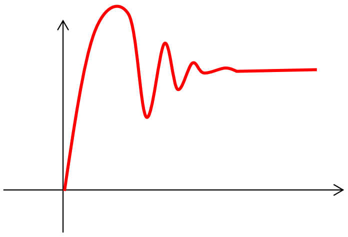


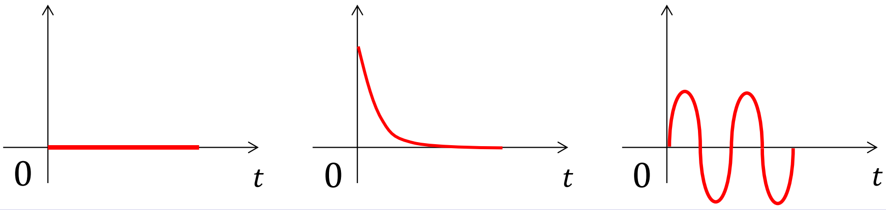
- Sine



# Transient Response and Steady-State Response

$$\boxed{c(t) = c_{\mathrm{tr}}(t) + c_{\mathrm{ss}}(t)} \Rightarrow v_c(t) = E\left(1 - e^{-t/RC}\right) = \underbrace{\boxed{E}}_{\text{Steady-state}} + \underbrace{\boxed{-Ee^{-t/RC}}}_{\text{Transient}}$$





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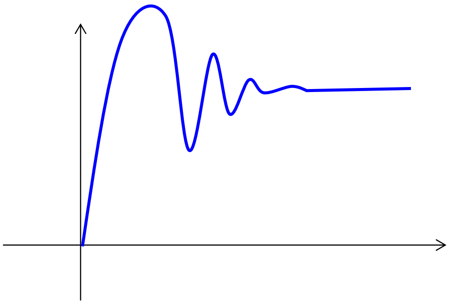
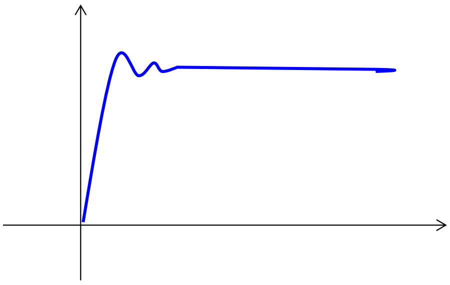
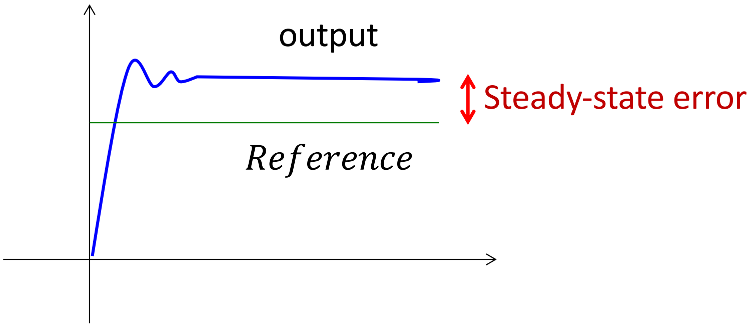
# Steady-State Error

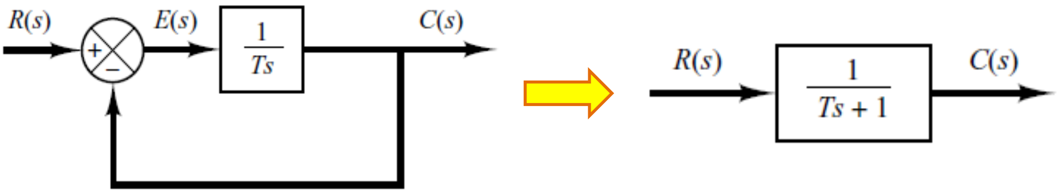






## First-Order Systems Response Analysis

$$\frac{C(s)}{R(s)}=\frac{1}{Ts+1}$$



- **Unit-Step Response of First-Order Systems**

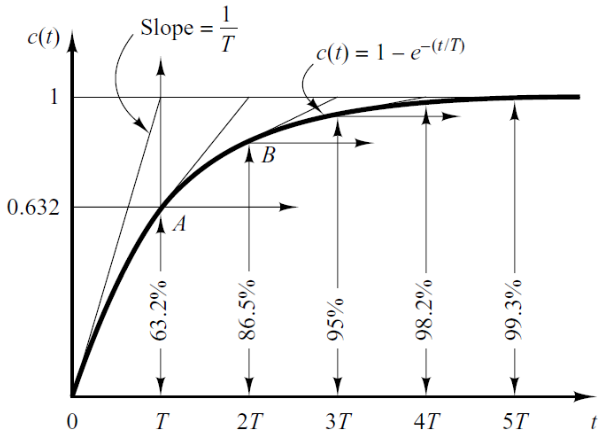
$R(s)=1/s$

$$C(s)=\frac{1}{Ts+1}\frac{1}{s}=\frac{1}{s}-\frac{T}{Ts+1}=\frac{1}{s}-\frac{1}{s+(1/T)}$$

$$\leftrightarrow c(t)=1-e^{-t/T}\quad for\ t\geq 0$$

$$\left.\frac{dc}{dt}\right|_{t=0}=\left.\frac{1}{T}e^{-t/T}\right|_{t=0}=\frac{1}{T}$$

$t\geq 4T$ error is about **2%**



Exponential response curve

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## Continue

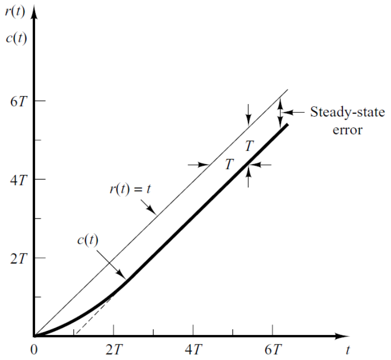
- **Unit-Ramp Response of First-Order Systems**

$R(s)=1/s^2$

$$C(s)=\frac{1}{Ts+1}\frac{1}{s^2}=\frac{1}{s^2}-\frac{T}{s}+\frac{T^2}{Ts+1}$$

$$\leftrightarrow c(t)=t-T+Te^{-\frac{t}{T}}\quad for\ t\geq 0$$

$$e(t)=r(t)-c(t)=T(1-e^{-t/T})\ ⇨\ e(\infty)=T$$



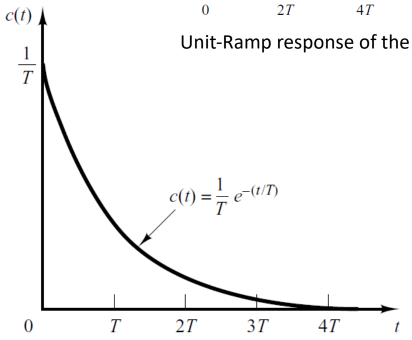
Unit-Ramp response of the previous system

- **Unit-Impulse Response of First-Order Systems**

$R(s)=1$

$$C(s)=\frac{1}{Ts+1}$$

$$\leftrightarrow c(t)=\frac{1}{T}e^{-t/T}\quad for\ t\geq 0$$



Unit-Impulse response

# An Important Property of LTI Systems

| | Input (s-domain) | output (s-domain) | output (time-domain) |
|---|---|---|---|
| Ramp | $\frac{1}{s^2}$ | $\frac{1}{s^2}\frac{1}{Ts+1}$ | $t - T + Te^{-t/T}$ |
| Step | $\frac{1}{s}$ | $\frac{1}{s}\frac{1}{Ts+1}$ | $1 - e^{-t/T}$ |
| Impulse | 1 | $\frac{1}{Ts+1}$ | $\frac{1}{T}e^{-t/T}$ |

differential

differential

- ○ The response to the derivative of an input signal can be obtained by differentiating the response of the system to the original signal.
- ○ It can also be seen that the response to the integral of the original signal can be obtained by integrating the response of the system to the original signal and by determining the integration constant from the zero-output initial condition.
- ○ This is a property of linear time-invariant systems.

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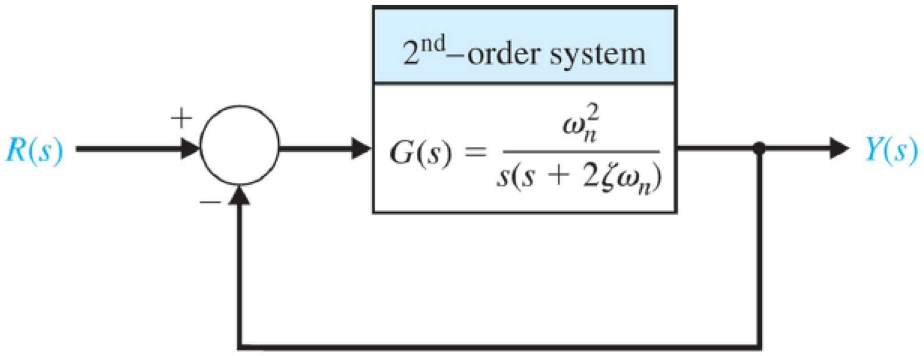
# Second-Order Systems



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

# Second-Order Systems

- Closed-loop transfer function (standard form)

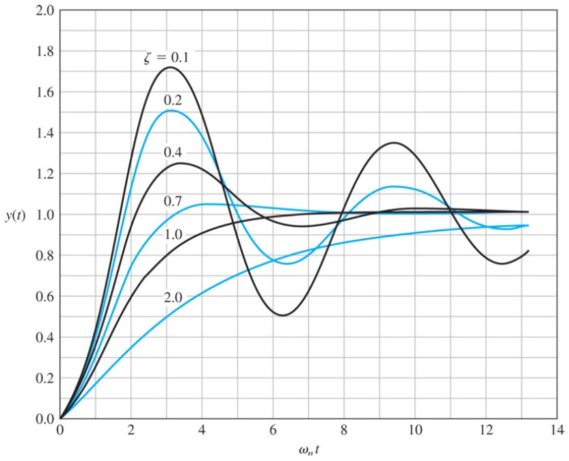
$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$ ⇨ $s = \omega_n(-\zeta \pm \sqrt{\zeta^2 - 1})$

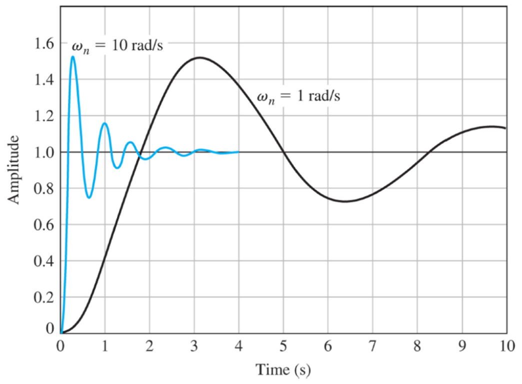
- $0 < \zeta < 1$ : *Underdamped case*
- $\zeta = 1$ : *Critically damped case*
- $\zeta > 1$ : *Overdamped case*

# Typical Step Response

Typical step responses, fixed $\omega_n$



Typical step responses, fixed $\zeta$



# Continue

$0 < \zeta < 1$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s+\zeta\omega_n+j\omega_d)(s+\zeta\omega_n-j\omega_d)}$$

*where* $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$\omega_d$ : *damped natural frequency*

- $R(s) = 1/s$

$$C(s) = \frac{\omega_n}{s^2+2\zeta\omega_n s+{\omega_n}^2}\frac{1}{s} = \frac{1}{s} - \frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+{\omega_n}^2}$$

$$= \frac{1}{s} - \frac{s+\zeta\omega_n}{\left(s+\zeta\omega_n\right)^2+{\omega_d}^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+{\omega_d}^2}$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

The frequency of transient oscillation is the $\omega_d$ damped natural frequency and thus varies with the damping ratio $\zeta$.

---

# Continue

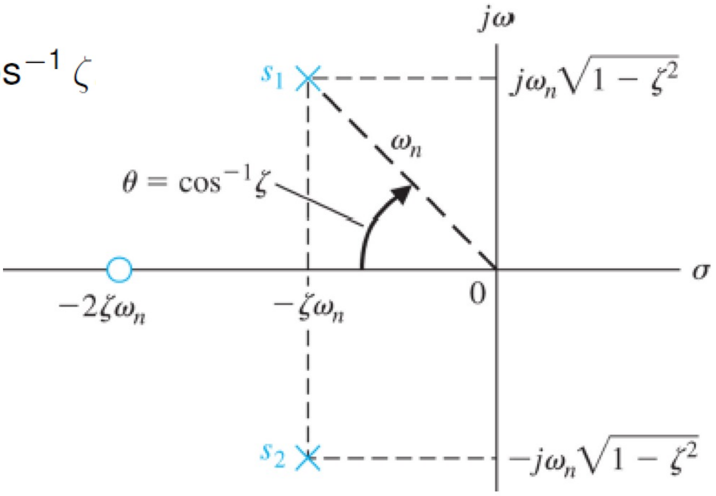
$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = 1 - \frac{1}{\beta}e^{-\zeta\omega_n t}\sin(\omega_n\beta t + \theta)$$

where $\beta = \sqrt{1-\zeta^2}$ and $\theta = \cos^{-1}\zeta$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$\Downarrow$$

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2})$$



## Continue

○ The error signal for this system

$$e(t) = r(t) - c(t) = 1 - \left\{1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right\}$$

$$= \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

○ $t = \infty$ ⇨ $e = 0$

○ $c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$

$\zeta = 0$ ⇨ $c(t) = 1 - \cos\omega_n t\,, \qquad for\ t \geq 0$

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## Continue …

**$\zeta = 1$**

$$\frac{C(s)}{R(s)} = \frac{{\omega_n}^2}{s^2 + 2\omega_n s + {\omega_n}^2} = \frac{{\omega_n}^2}{(s+\omega_n)^2}$$

$R(s) = 1/s$ ⇨ $C(s) = \frac{{\omega_n}^2}{s(s+\omega_n)^2} \leftrightarrow c(t) = 1 - e^{-\omega_n t}(1+\omega_n t), \qquad t \geq 0$

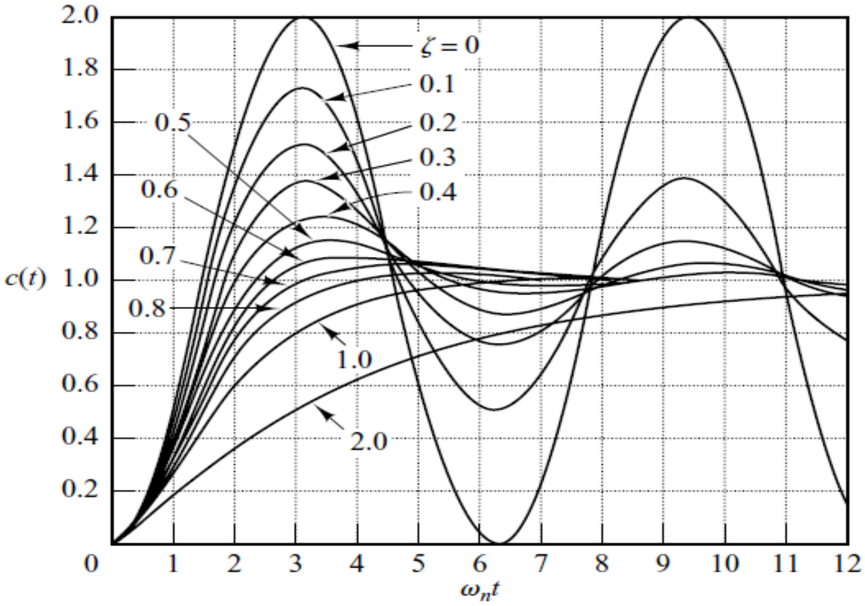
**$\zeta > 1$** ⇨ $C(s) = \frac{{\omega_n}^2}{\left(s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}\right)\left(s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}\right)s}$

$$c(t) = 1 + \frac{1}{2\sqrt{\zeta^2-1}(\zeta+\sqrt{\zeta^2-1})}e^{-\left(\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} - \frac{1}{2\sqrt{\zeta^2-1}(\zeta+\sqrt{\zeta^2-1})}e^{-\left(\zeta+\sqrt{\zeta^2-1}\right)\omega_n t}$$

$$= 1 + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-s_1\,t}}{s_1} - \frac{e^{-s_2\,t}}{s_2}\right)$$

$where\ s_1 = \left(\zeta+\sqrt{\zeta^2-1}\right)\omega_n,\ s_2 = (\zeta-\sqrt{\zeta^2-1})\omega_n$

## Step Response of Second-Order System



p.169 (Figure5-7): Unit-step response curves of the system shown in Figure 5-6.

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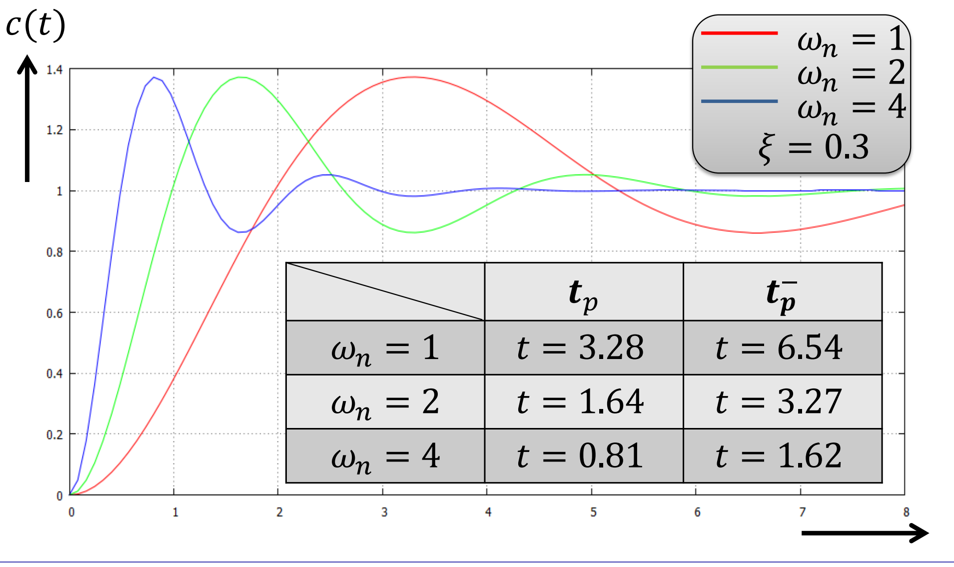
## Continue

A secondary system for different $\omega_n$and the same $\xi$ almost show the same overshoot and vibration pattern. They have the same relative stability.

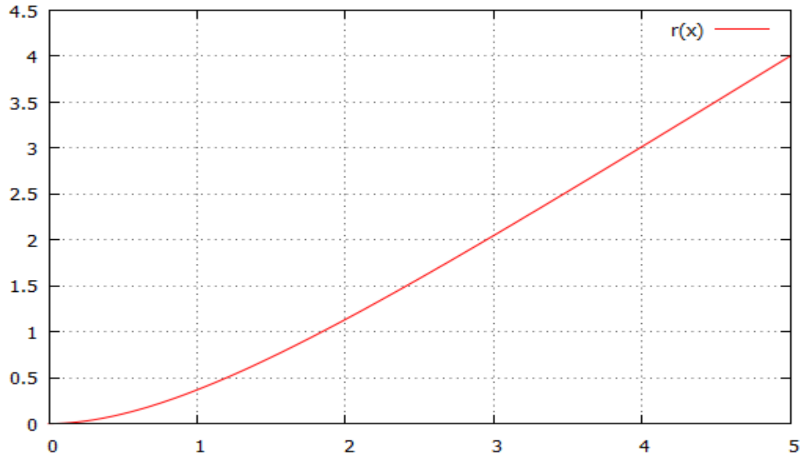


| | $\boldsymbol{t_p}$ | $\boldsymbol{t_p^-}$ |
|---|---|---|
| $\omega_n = 1$ | $t = 3.28$ | $t = 6.54$ |
| $\omega_n = 2$ | $t = 1.64$ | $t = 3.27$ |
| $\omega_n = 4$ | $t = 0.81$ | $t = 1.62$ |

# Example

$$\frac{C(s)}{R(s)} = \frac{1}{s(s+1)} \quad \leftrightarrow \quad C(s) = \frac{1}{s^2(s+1)} = -\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1}$$

$$\leftrightarrow \quad c(t) = -1 + t + e^{-t}$$



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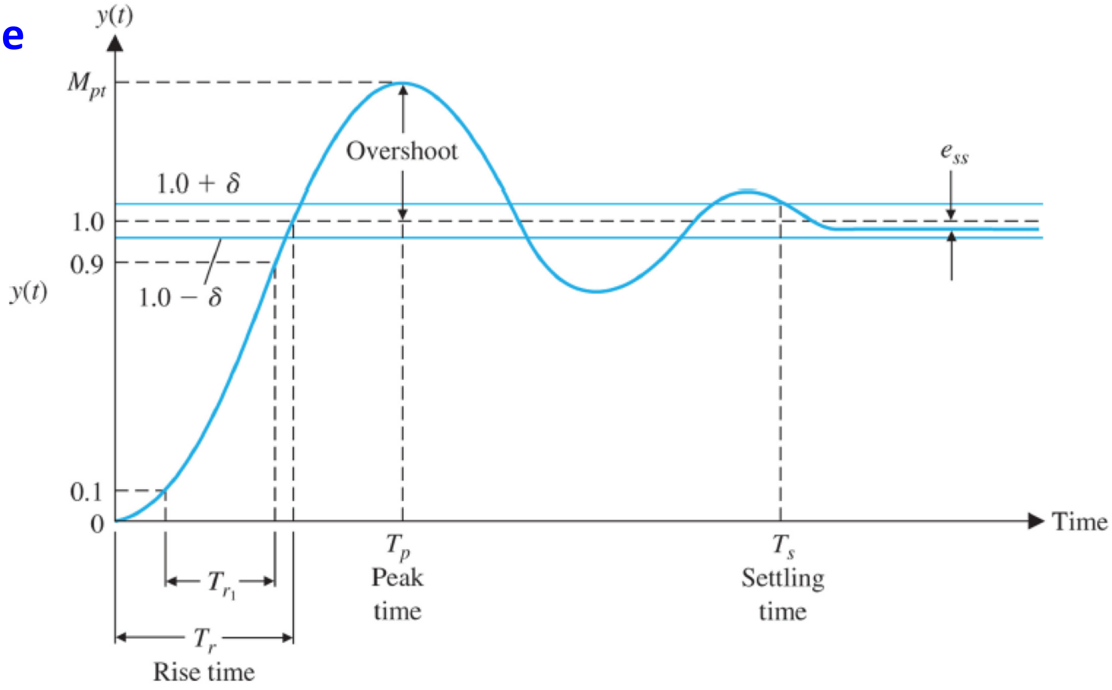
# Transient-Response Specifications

- ○ **Performance Indices**



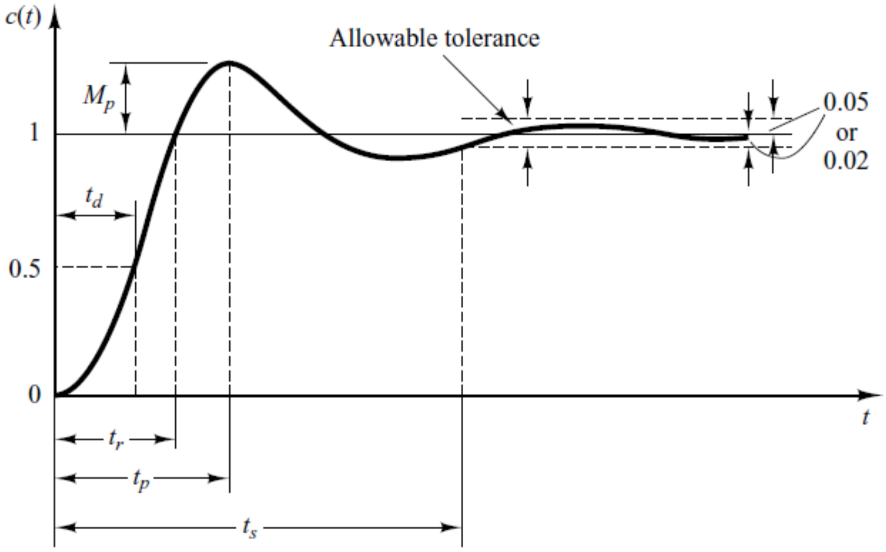
# Transient-Response Specifications

○ **Delay time:** $t_d$

The time required for the response to reach half the final value the very first time.

○ **Rise time:** $t_r$

The time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value.



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# Transient-Response Specifications

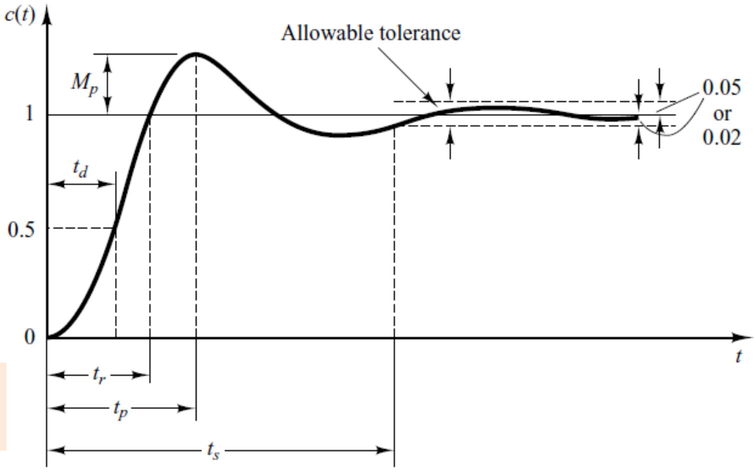
○ **Peak time:** $t_p$

The time required for the response to reach the first peak of the overshoot.

○ **Maximum overshoot:** $M_p$

The maximum peak value of the response curve measured from unity.

Max. percent overshoot= $\frac{c(t_p)-c(\infty)}{c(\infty)}\times 100\%$



○ **Settling time:** $t_s$

The time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%).

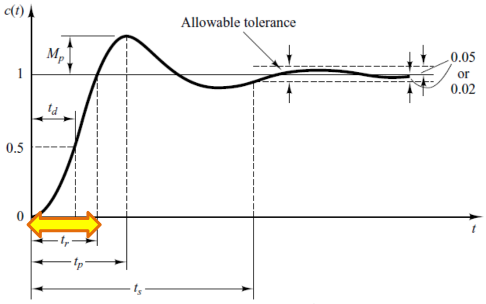
## 2nd-Order Systems and Transient-Response Specifications

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

⇩

$$\mathcal{L}^{-1}[C(s)] = c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

$$\boxed{\xi\omega_n = \sigma \ ;. \quad \omega_d = \omega_n\sqrt{1-\zeta^2}}$$

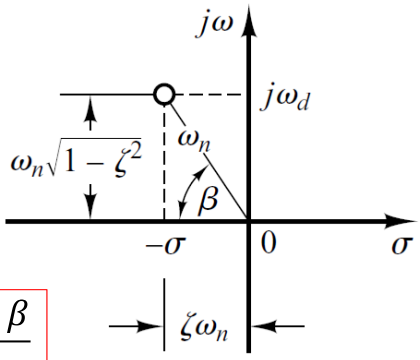


### ○ Rise time $t_r$

⇨ $$c(t_r) = 1 = 1 - e^{-\zeta\omega_n t_r}\left(\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r\right)$$

$$\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r = 0 \quad (\ e^{-\zeta\omega_n t_r} \neq 0)$$

⇨ $$\tan\omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta} = -\frac{\omega_d}{\sigma}$$ ⇨ $$\boxed{t_r = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{-\sigma}\right) = \frac{\pi-\beta}{\omega_d}}$$



Definition of the angle $\beta$

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## Continue

### ○ Peak time $t_p$

$$c(t) = 1 - e^{-\zeta\omega_n t}(cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} sin\omega_d t)$$

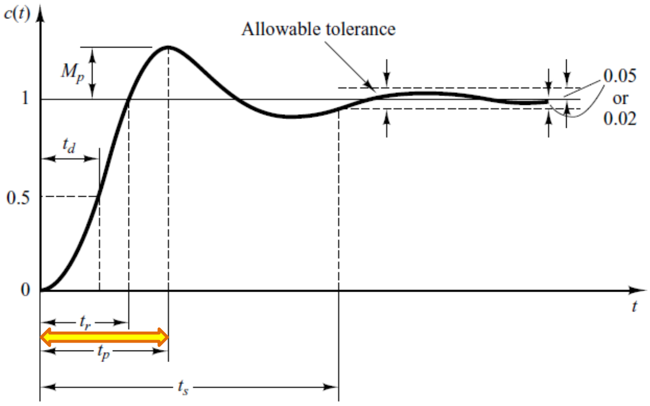
$$\frac{dc}{dt} = \zeta\omega_n e^{-\zeta\omega_n t}\left(cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} sin\omega_d t\right) + e^{-\zeta\omega_n t}(\omega_d sin\omega_d t - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}} cos\omega_d t)$$

at $t$=$t_p$: $\frac{dc}{dt}$=0

$$\frac{dc}{dt}|_{t=t_p} = (sin\omega_d t_p)\ \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_p} = 0$$

⇨ $sin\omega_d t_p = 0$ ⇨ $\omega_d t_p$=0, π, 2π, 3π, …

⇨ $\omega_d t_p$=π ⇨ $\boxed{t_p = \frac{\pi}{\omega_d}}$



## Continue

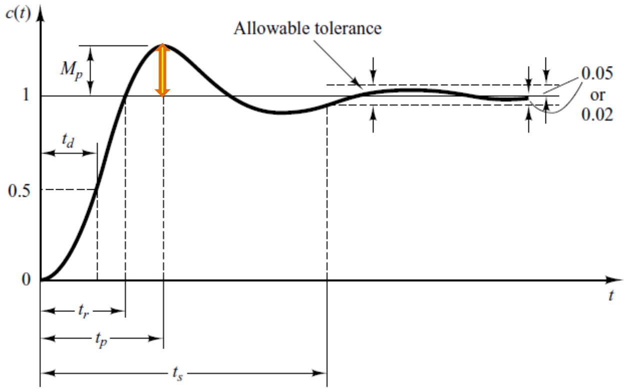
$$c(t) = 1 - e^{-\zeta\omega_n t}(cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} sin\omega_d t)$$

- ○ **Maximum overshoot $M_p$** ： $t{=}t_p{=}\pi/\omega_d$

$$M_p = c(t_p) - 1$$

$$= -e^{-\zeta\omega_n\left(\frac{\pi}{\omega_d}\right)}\left(cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}} sin\pi\right)$$

$$\boxed{= e^{-(\sigma/\omega_d)\pi} = e^{-(\frac{\zeta}{\sqrt{1-\zeta^2}})\pi}}$$



The maximum percent overshoot is $e^{-(\sigma/\omega_d)\pi} \times 100\%$

If $c(\infty) \neq 1$: ⇒ $M_p = \frac{c(t_p) - c(\infty)}{c(\infty)}$

---

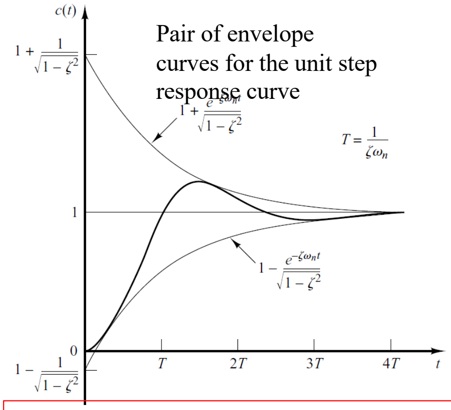
## Continue

- ○ **Settling time $t_s$**

For an underdamped second-order system, the transient response is

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} sin(\omega_d t + tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta})$$

- $1 \pm \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$ are the envelope curves of the transient response to a unit-step input
- The settling time corresponding to a 2% or 5% tolerance band may be measured in terms of the time constant $T = \frac{1}{\zeta\omega_n}$



- ±2% criterion :

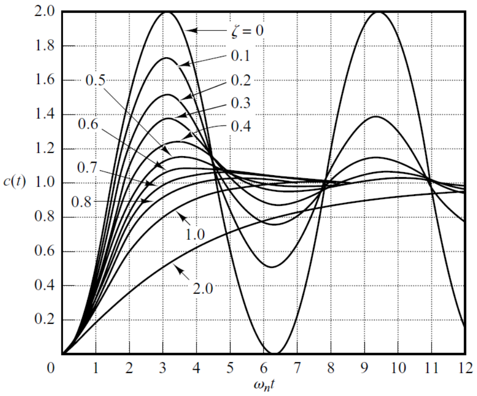
$$t_s = 4T = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n}$$

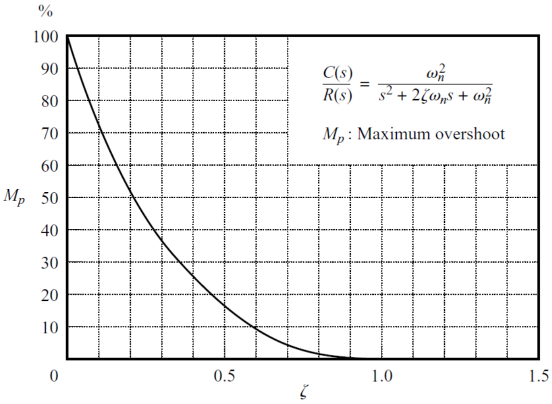
- ±5% criterion :

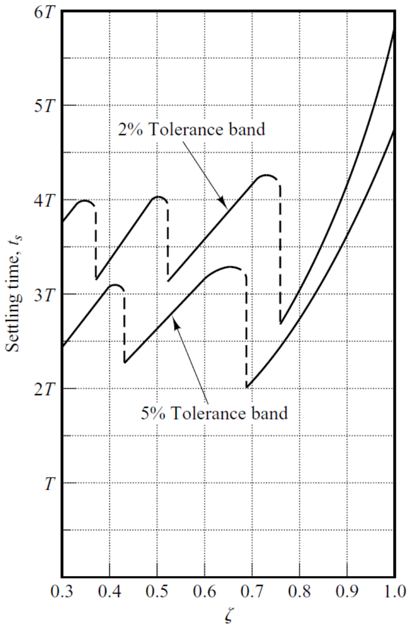
$$t_s = 3T = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n}$$

# Continue

- $M_p$ and $t_s$ versus $\zeta$







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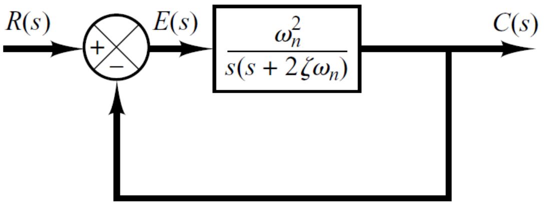
# Example

The below system is subjected to a unit-step input, find the transient characteristics for $\zeta$=0.6, and $\omega_n$=5



Solution: $\omega_d = \omega_n\sqrt{1-\zeta^2} = 4$、$\sigma = \zeta\omega_n = 3$

$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{3.14 - \beta}{\omega_d}$$

$$\beta = tan^{-1}\frac{\omega_d}{\sigma} = tan^{-1}\frac{4}{3} = 0.93\, rad$$

$$t_r = \frac{3.14 - 0.93}{4} = 0.55\, sec$$

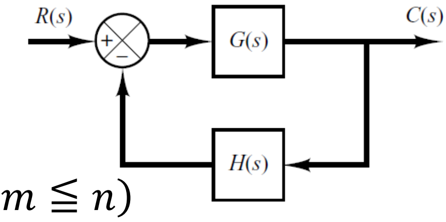
$$t_p = \frac{\pi}{\omega_d} = \frac{3.14}{4} = 0.785\, sec$$

$$M_p = e^{-(\sigma/\omega_d)\pi} = e^{-\left(\frac{3}{4}\right)\times 3.14} = 0.095$$

$t_s$ (2%): $t_s = \frac{4}{\sigma} = \frac{4}{3} = 1.33\, sec$

$t_s$ (5%): $t_s = \frac{3}{\sigma} = \frac{3}{3} = 1\, sec$

## Transient Response of Higher-Order Systems



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} \Rightarrow G(s) = \frac{p(s)}{q(s)} \quad H(s) = \frac{n(s)}{d(s)}$$

$$\frac{C(s)}{R(s)} = \frac{p(s)d(s)}{q(s)d(s)+p(s)n(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} \quad (m \leqq n)$$

- ○ MATLAB may be used for finding the roots of the denominator polynomial. Use the command *roots*(*den*). $\frac{C(s)}{R(s)} = \frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$
- ○ Consider first the case where the closed-loop poles are all real and distinct

$$C(s) = \frac{a}{s} + \sum_{i=1}^{n} \frac{a_i}{s+p_i}$$

- ○ The partial-fraction expansion of C(s), can be obtained easily with MATLAB, using the *residue* command.

---

## Continue

- ○ Next, consider the case where the poles of C(s) consist of real poles and pairs of complex-conjugate poles:

$$C(s) = \frac{a}{s} + \sum_{j=1}^{q} \frac{a_j}{s+p_j} + \sum_{k=1}^{r} \frac{b_k(s+\zeta_k\omega_k) + c_k\omega_k\sqrt{1-\zeta_k^2}}{s^2+2\zeta_k\omega_k s+\omega_k^2}$$

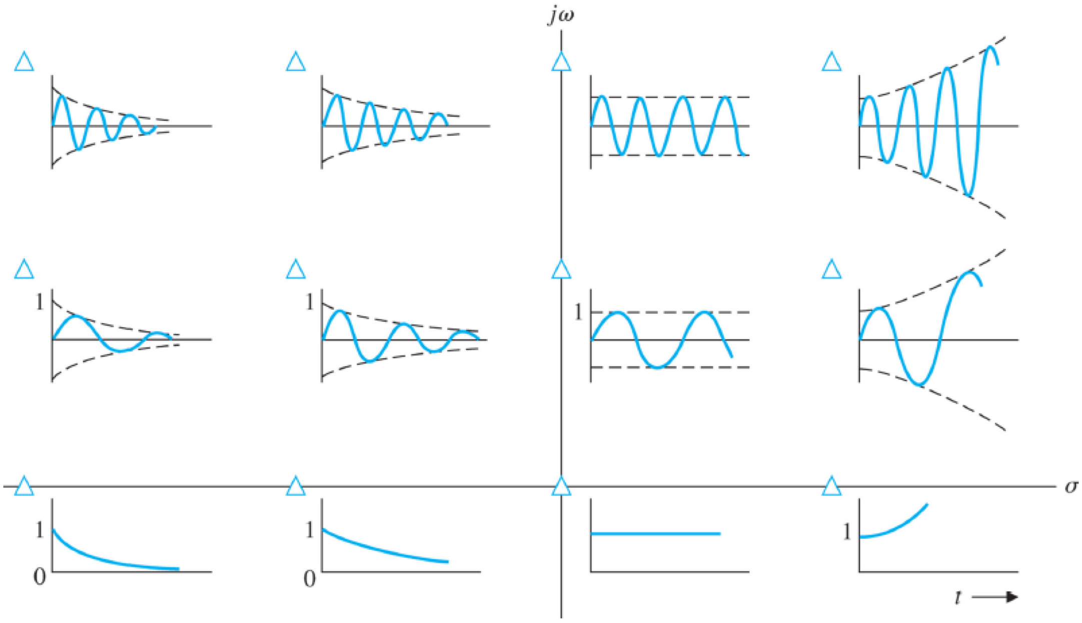
$$(q+2r=n)$$

$$\Downarrow$$

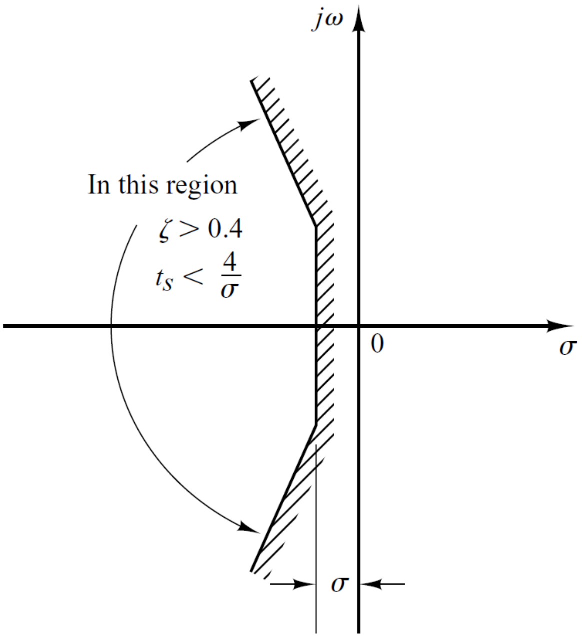
$$c(t) = a + \sum_{j=1}^{q} a_j e^{-p_i t} + \sum_{k=1}^{r} b_k e^{-\zeta_k\omega_k t} \cos\omega_k\sqrt{1-\zeta_k^2}\, t$$

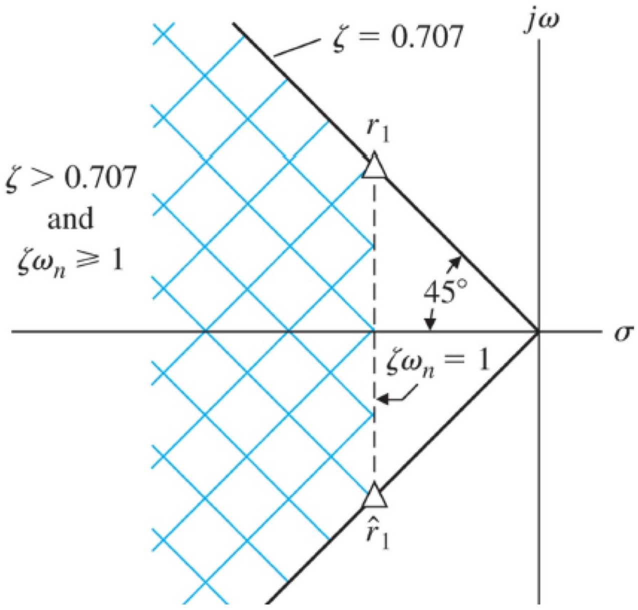
$$+ \sum_{k=1}^{r} c_k e^{-\zeta_k\omega_k t} \sin\omega_k\sqrt{1-\zeta_k^2}\, t \quad , \text{ for } t \geqq 0$$

# A 2nd-Order System Response



# Dominant Closed-Loop Poles





# Thank You!