Linear Control System

Steady-State Error and Stability

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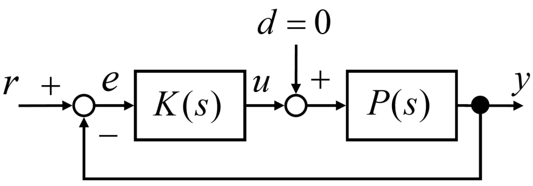
- Stability

Error in Feedback Systems

$$e(t) = r(t) - y(t)$$
$$L(s) = P(s)K(s)$$
$$y(s) = L(s)e(s)$$



$$e(s) = r(s) - L(s)e(s) \Rightarrow (1+L(s))e(s) = r(s) \Rightarrow e(s) = \frac{1}{1+L(s)} r(s)$$

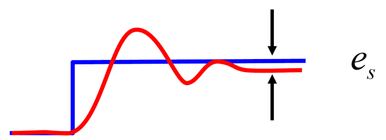
$$e_s = \lim_{t\to\infty} e(t) = \lim_{s\to 0} s\cdot e(s) = \lim_{s\to 0} s\frac{1}{1+L(s)} r(s)$$

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \qquad F(s) := \int_0^\infty f(t)e^{-st}dt$$

Steady-State Error

$r(t)=1 \quad \left(r(s)=\frac{1}{s}\right) \Rightarrow$

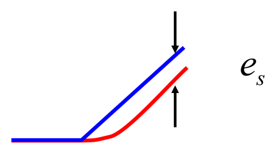
$$e_s = \lim_{s\to 0} s\frac{1}{1+L(s)}\cdot\frac{1}{s} = \frac{1}{1+\lim_{s\to 0} L(s)}$$



$$K_p = \lim_{s\to 0} L(s) = L(0)$$

$r(t)=t \quad \left(r(s)=\frac{1}{s^2}\right) \Rightarrow$

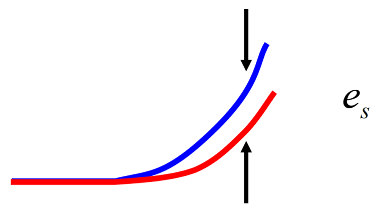
$$e_s = \lim_{s\to 0} s\frac{1}{1+L(s)}\cdot\frac{1}{s^2} = \lim_{s\to 0}\frac{1}{sL(s)}$$



$$K_v = \lim_{s\to 0} sL(s)$$

$r(t)=\frac{1}{2}t^2 \quad r(s) = \frac{1}{s^3} \Rightarrow$

$$e_s = \lim_{s\to 0}\frac{1}{s^2 L(s)}$$

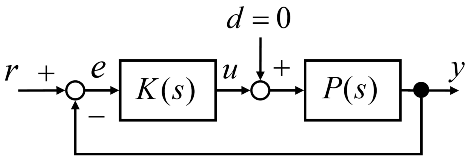


$$K_a = \lim_{s\to 0} s^2 L(s)$$

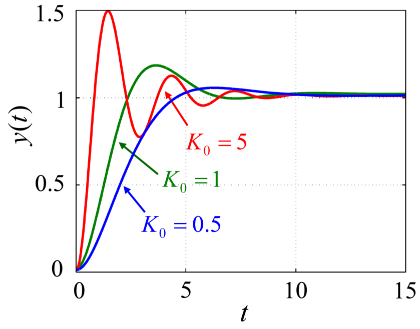
Continue

$$L(s)=P(s)K(s)=\frac{K_0}{s(s+1)}$$

$$P(s)=\frac{1}{s},\quad K(s)=\frac{K_0}{s+1}\ (K_0>0)$$

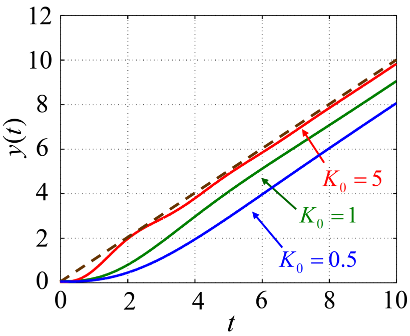


$$K_p=L(0)=\infty \Rightarrow e_s=\frac{1}{1+\lim_{s\to0}L(s)}=\frac{1}{1+L(0)}=\frac{1}{1+K_p}=0$$



$$K_v=\lim_{s\to0}sL(s)=\lim_{s\to0}s\frac{K_0}{s(s+1)}=K_0$$

$$\Rightarrow e_s=\lim_{s\to0}\frac{1}{sL(s)}=\frac{1}{K_0}$$



System Type

$$L(s)=\frac{b_ms^m+b_{m-1}s^{m-1}+\cdots+b_0}{s^l(s^n+a_{n-1}s^{n-1}+\cdots+a_0)}$$

$l=1 \Rightarrow K_p=L(0)=\infty \Rightarrow \frac{1}{1+K_p}=0$

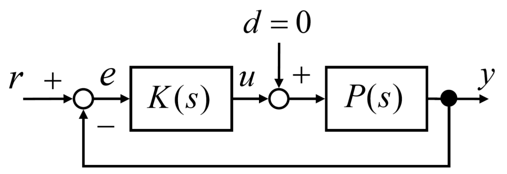
$l=2 \Rightarrow K_v=\lim_{s\to0}s\cdot L(s)=\infty \Rightarrow \frac{1}{K_v}=0$

$l=3 \Rightarrow K_a=\lim_{s\to0}s^2\cdot L(s)=\infty \Rightarrow \frac{1}{K_a}=0$

Type \ $r(t)$	$r(t)=1$	$r(t)=t$	$r(t)=\frac{t^2}{2}$
0	$\frac{1}{1+K_p}$	∞	∞
1	0	$\frac{1}{K_v}$	∞
2	0	0	$\frac{1}{K_a}$

System Type

$$e(t)=r(t)-y(t) \Rightarrow e(s)=\frac{1}{1+L(s)}r(s)$$



$$L(s)=P(s)K(s)$$

$$e_s=\lim_{t\to\infty}e(t)=\lim_{s\to 0}s\cdot e(s)=\lim_{s\to 0}s\frac{1}{1+L(s)}r(s)$$

$$K_p=\lim_{s\to 0}L(s)=L(0)$$

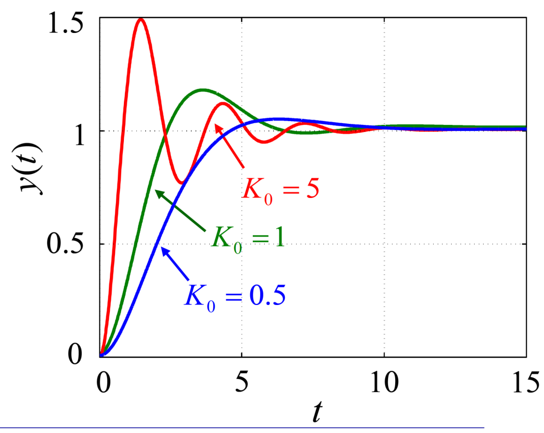
$$K_v=\lim_{s\to 0}sL(s)$$

$$K_a=\lim_{s\to 0}s^2L(s)$$

Example: $P(s)=\dfrac{1}{s}$, $K(s)=\dfrac{K_0}{s+1}$ $\quad(K_0>0)$

$$L(s)=P(s)K(s)=\frac{K_0}{s(s+1)} \qquad K_p=L(0)=\infty$$

$$\Rightarrow e_s=\frac{1}{1+\lim_{s\to 0}L(s)}=\frac{1}{1+L(0)}=\frac{1}{1+K_p}=0$$

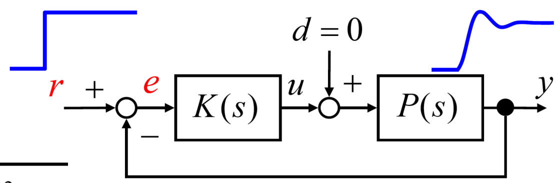


Summary: Performance Evaluation

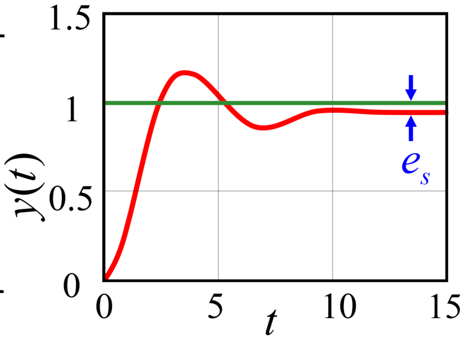
- Transient characteristics
- **Steady-state characteristics**

$$L(s)=P(s)K(s)$$

$$e_s=\frac{1}{1+L(0)} \qquad K_p=L(0)$$

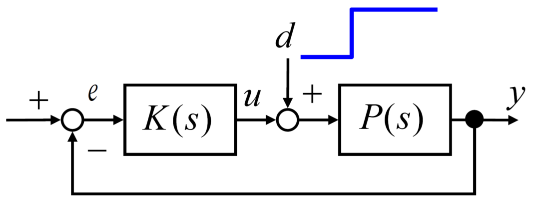


System type	$r(t)=1$	$r(t)=t$	$r(t)=t^2/2$
0 Type	$\frac{1}{1+K_p}$	∞	∞
1 Type	0	$\frac{1}{K_v}$	∞
2 Type	0	0	$\frac{1}{K_a}$



System Response

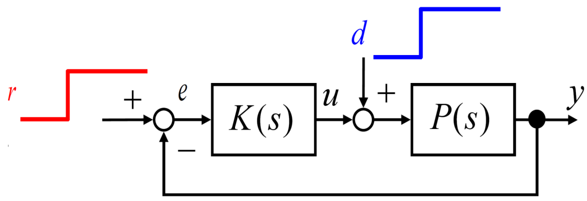
○ **Zero-Input Response:** $r = 0$ $d(s) = \frac{1}{s}$



$$e(s) = -y(s) = -\frac{P(s)}{1+L(s)}d(s)$$

$$\Rightarrow \lim_{t\to\infty} e(t) = -\lim_{t\to\infty} y(t) = -\lim_{s\to 0} sy(s) = -\frac{P(0)}{1+L(0)}$$

○ **Non-Zero Input and Disturbance Response:**



$$e(s) = \frac{1}{1+L(s)}r(s) - \frac{P(s)}{1+L(s)}d(s)$$

$$\Rightarrow \lim_{t\to\infty} e(t) = \frac{1}{1+L(0)} - \frac{P(0)}{1+L(0)}$$

Pole and Zero

impulse response transfer function: $G(s)$

○ **Real pole:** $-\sigma_i (i = 1 \sim M)$

Example: $G(s) = \frac{1}{s+1} \Rightarrow \sigma = -1$

○ **Complex conjugate pole:** $-\alpha_i \pm j\omega_i (i = 1 \sim N)$

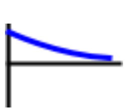
Example: $G(s) = \frac{1}{s^2+s+1} \Rightarrow \frac{-1 \pm j\sqrt{3}}{2} \Rightarrow \alpha = \frac{-1}{2}, \quad \omega = \frac{\sqrt{3}}{2}$

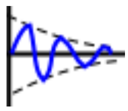
System (Impulse) Response

$$y(s) = G(s) = \sum_{i=1}^{M} \frac{A_i}{s+\sigma_i} + \sum_{i=1}^{N} \frac{B_i}{(s+\alpha_i)^2 + \omega_i^2}$$

- **Impulse response**

$$y(t) = \sum_{i=1}^{M} A_i e^{-\sigma_i t} + \sum_{i=1}^{N} \frac{B_i}{\omega_i} e^{-\alpha_i t} \sin \omega_i t$$



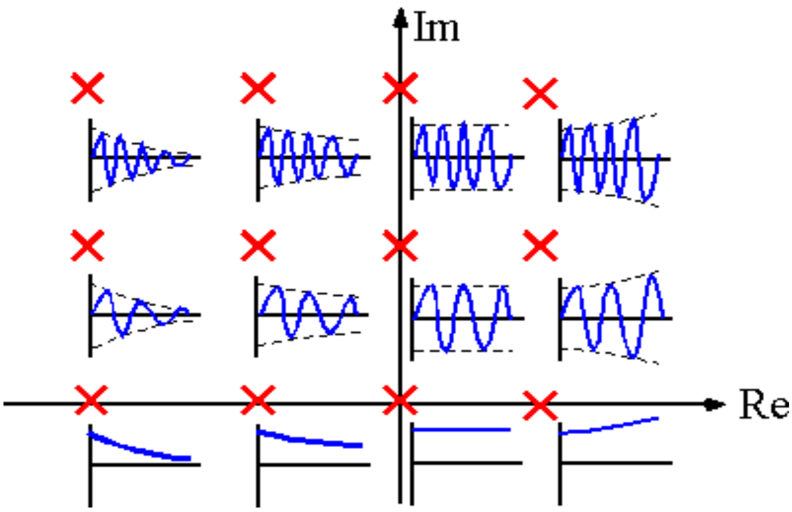


Example: $G(s) = \frac{1}{s^2+s+1} = \frac{1}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}} = \frac{1}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

Poles and System Response

- The size of the real part of the pole: ⇨ Convergence speed
- The size of the imaginary part of the pole: ⇨ Period of oscillation component

Pole position and impulse response



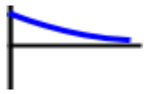
System (Step) Response

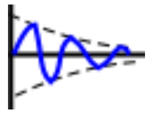
- Step response (Laplace transform) from partial fraction decomposition:

$$y(s) = G(s)\frac{1}{s} = \frac{A_0}{s} + \sum_{i=1}^{M} \frac{A_i}{s+\sigma_i} + \sum_{i=1}^{N} \frac{B_i}{(s+\alpha_i)^2 + \omega_i^2}$$

- Step response

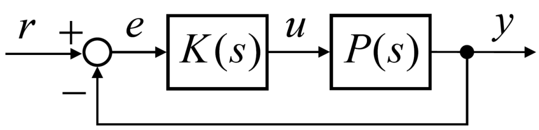
$$y(t) = \underline{A_0} + \sum_{i=1}^{M} \underline{A_i e^{-\sigma_i t}} + \sum_{i=1}^{N} \frac{B_i}{\omega_i} \underline{e^{-\alpha_i t} \sin \omega_i t}$$





Zeros and Poles

$$S = \frac{1}{1+PK}, \quad T = \frac{PK}{1+PK}$$



$$S(s) = \frac{1}{1+P(s)K(s)}$$

$S(p) = 0 \Leftarrow P(p) = \infty$

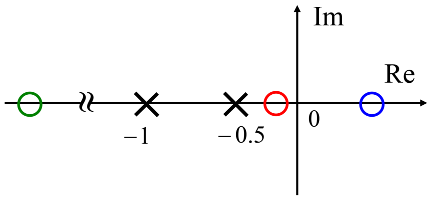
$$S(p) = \frac{1}{1+P(p)K(p)} = 0$$

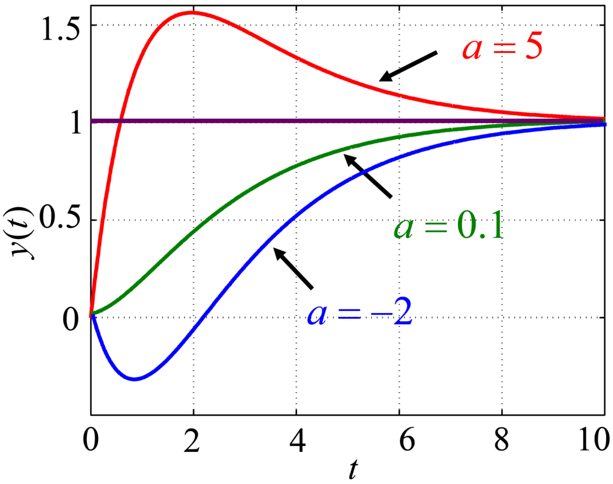
$S(z) = 1 \Leftarrow P(z) = 0$

$$S(z) = \frac{1}{1+P(z)K(z)} = \frac{1}{1} = 1$$

Impact of Zeros

Example: $G(s) = \dfrac{as+1}{(s+1)(2s+1)}$ Poles: $-1, \ -0.5$ Zero: $-\dfrac{1}{a}$





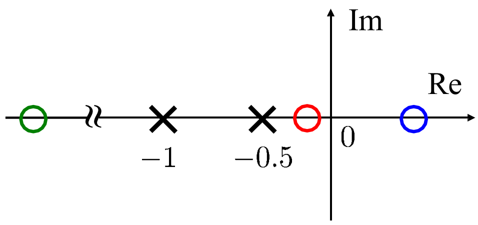
a : Small ➡ no effect

a : Big ➡ overshoot

$a < 0$: Unstable ➡ non-minimum phase

Effect of zero

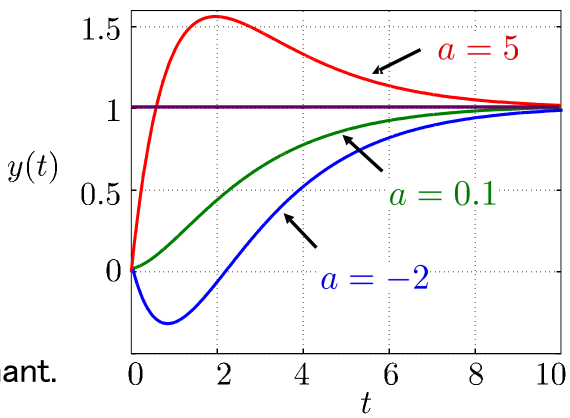
- Example: $G(s) = \dfrac{as+1}{(s+1)(2s+1)}$ Poles: $-1, \ -0.5$ Zero: $-\dfrac{1}{a}$



$a < 0$ ➡ Non-minimum phase

a small ➡ No effect

a big ➡ Overshoot

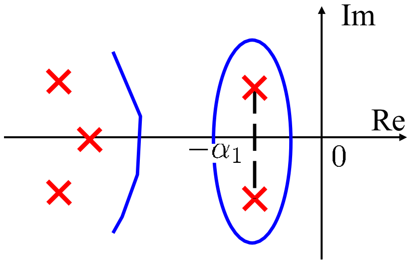


- The response of the pole near the origin is the dominant.

Dominant pole

$$y(t) = A_0 + \sum_{i=1}^{M} A_i e^{-\sigma_i t} + \sum_{i=1}^{N} \frac{B_i}{\omega_i} e^{-\alpha_i t} \sin \omega_i t$$

$$\begin{cases} 0 < \alpha_1 \ll \alpha_j \quad (j = 2 \sim N) \\ 0 < \alpha_1 \ll \sigma_j \quad (j = 1 \sim M) \end{cases}$$

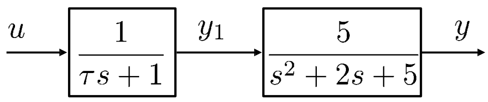


$e^{-\alpha_j t}$, $e^{-\sigma_j t}$ Decreases rapidly

⇨ The slowest mode $e^{-\alpha_1 t}$ Is dominant ⇨ $y(t) \approx A_0 + \frac{B_1}{\omega_1} e^{-\alpha_1 t} \sin \omega_1 t$

Example

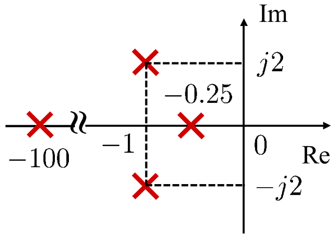
$$G(s) = G_1(s)G_2(s) = \frac{1}{\tau s + 1} \cdot \frac{5}{s^2 + 2s + 5}$$

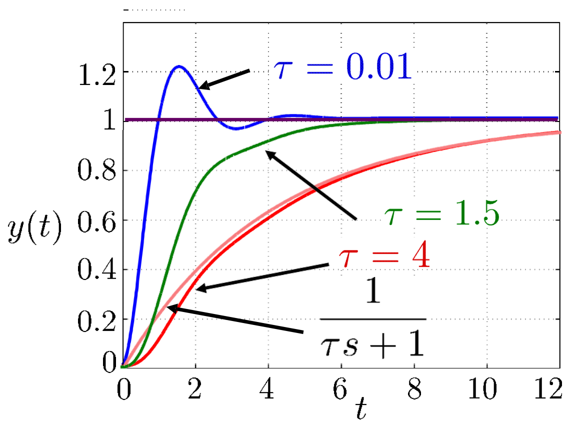


$$\begin{cases} \tau s + 1 = 0 \Rightarrow s = -\frac{1}{\tau} \\ s^2 + 2s + 5 = 0 \Rightarrow s = -1 \pm j2 \end{cases}$$

$\tau = 0.01 \quad -\frac{1}{\tau} = -100 \quad G \approx G_2$

$\tau = 4 \quad -\frac{1}{\tau} = -0.25 \quad G \approx G_1$

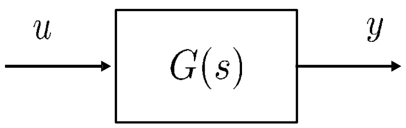




Stability: BIBO Stability

Stability: A causal, linear, time-invariant system is stable if and only if all its poles have negative real parts.

- Bounded input bounded output (BIBO) stability



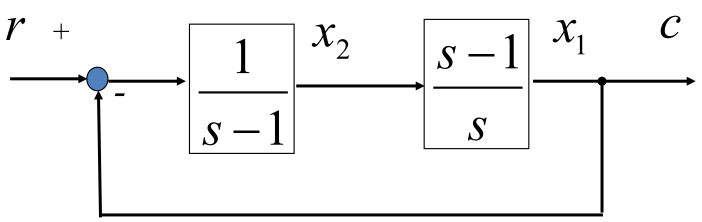
$$|u(t)| < \infty \Rightarrow |y(t)| < \infty$$

or

$$\int_0^{\infty} |g(t)|\, dt < \infty$$

Example

Check the BIBO and internal stability of the following system:

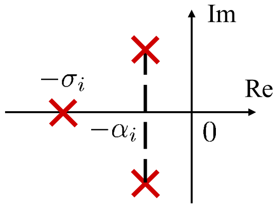


$$\frac{c(s)}{r(s)} = \frac{\frac{1}{s}}{1+\frac{1}{s}} = \frac{1}{s+1} \Rightarrow p=-1 \Rightarrow \text{BIBO stability}$$

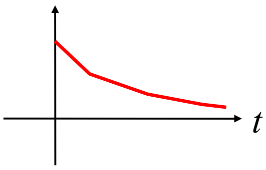
Step response (stable system)

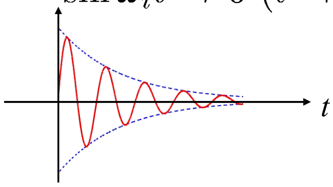
$$y(t) = A_0 + \sum_{i=1}^{M} A_i e^{-\sigma_i t} + \sum_{i=1}^{N} \frac{B_i}{\omega_i} e^{-\alpha_i t} \sin \omega_i t$$

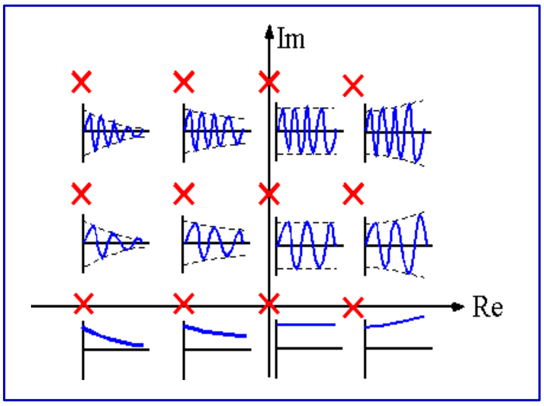
$$-\sigma_i < 0, \quad -\alpha_i < 0$$



$$e^{-\sigma_i t} \to 0 \ (t \to \infty) \qquad e^{-\alpha_i t} \sin \omega_i t \to 0 \ (t \to \infty)$$

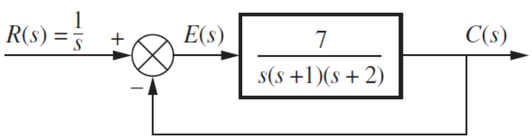




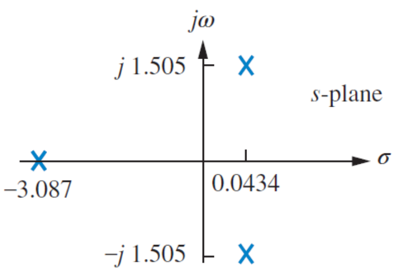


Step response (unstable system)

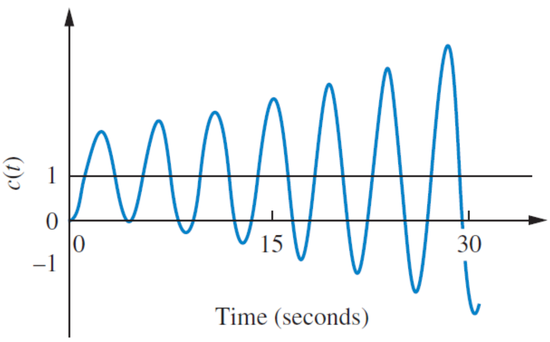
$$y(t) = A_0 + \sum_{i=1}^{M} A_i e^{-\sigma_i t} + \sum_{i=1}^{N} \frac{B_i}{\omega_i} e^{-\alpha_i t} \sin \omega_i t$$



Unstable system



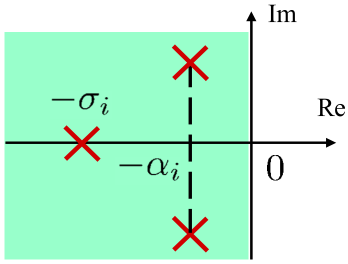
Unstable system's closed-loop poles (not to scale)



Necessary and sufficient conditions for stability

(A) Negative real parts of all poles

$$D(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \quad (a_n > 0)$$
$$= a_n \prod_{i=1}^{M}(s+\sigma_i)\prod_{i=1}^{N}\left(s^2 + 2\alpha_i s + (\alpha_i^2 + \omega_i^2)\right)$$

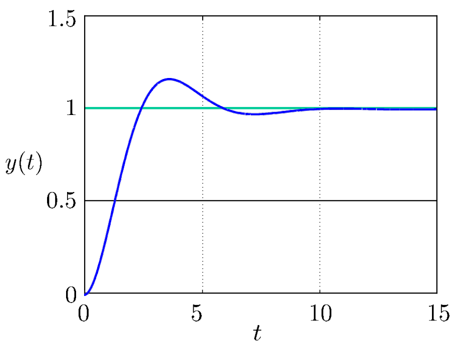


$$\sigma_i > 0,\ \alpha_i > 0$$

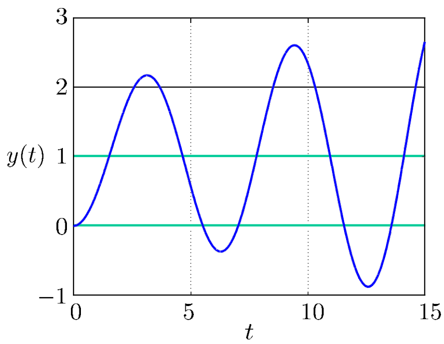
(B) All coefficients of $a_n,\ a_{n-1},\ \cdots, a_0$ must have same sign

Example

$$G_a(s) = \frac{1}{s^2 + s + 1}$$



$$G_b(s) = \frac{1}{s^2 - 0.1s + 1}$$



Example

$$G_1(s) = \frac{N_1(s)}{D_1(s)} \quad G_2(s) = \frac{N_2(s)}{D_2(s)}$$

$$D_1(s) = s^5 + s^4 + 3s^3 + 2s^2 + 6s + 2$$

$+1 \quad +1 \quad +3 \quad +2 \quad +6 \quad +2$ (B: OK)

$$D_2(s) = s^5 + s^4 + 6s^3 + 3s^2 + 4s + 1$$ (B: OK)

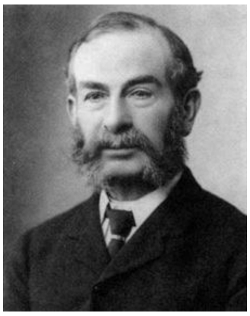
$+1 \quad +1 \quad +6 \quad +3 \quad +4 \quad +1$

- Are they both stable?

$D_1(s) = 0 \Rightarrow \underline{0.56 \pm 1.37j}, \; -0.89 \pm 1.33j, \; -0.35$ unstable

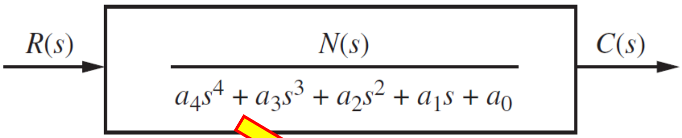
$D_2(s) = 0 \Rightarrow -0.26 \pm 2.21j, \; -0.10 \pm 0.85j, \; -0.28$ stable

Routh's method



- Using the "Routh Table" (1905). Then it is generalized to Routh-Hurwitz method

Example:



Routh Table:

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

Routh's stability criterion

○ **Completing the "Routh Table":** $D(s) = a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$

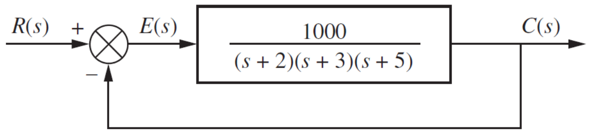
s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

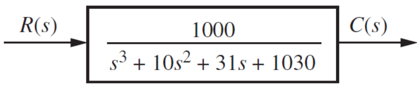
Routh's stability criterion

Theorem 1:

The number of sign changes in the first column of the Routh table equals the number of roots of the polynomial in the *Right Half-Plane* (RHP).

Example





s^3	1	31	0
s^2	~~10~~ 1	~~1030~~ 103	0
s^1	$\dfrac{-\begin{vmatrix} 1 & 31 \\ 1 & 103 \end{vmatrix}}{1} = -72$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}}{1} = 0$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$
s^0	$\dfrac{-\begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix}}{-72} = 103$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$

- **Unstable with 2 roots at the RHP**

Special Cases

- **Case 1: Zero Only in the First Column**

  Solution 1: via Epsilon method

  Solution 2: via Reverse coefficients

- **Case 2: Entire Row is Zero**

  Solution: via an Auxiliary Polynomial for above row of zeros

Case 1: Solution 1

Example: $$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

s^5	1	3	5
s^4	2	6	3
s^3	0	$\frac{7}{2}$	0
s^2	$\frac{6\epsilon - 7}{\epsilon}$	3	0
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	0
s^0	3	0	0

Continue

Example: $$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

				$\epsilon = +$	$\epsilon = -$
s^5	1	3	5	+	+
s^4	2	6	3	+	+
s^3	~~0~~ ϵ	$\frac{7}{2}$	0	+	−
s^2	$\frac{6\epsilon - 7}{\epsilon}$	3	0	−	+
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	0	+	+
s^0	3	0	0	+	+

- **Unstable with 2 roots at the RHP**

Case 1: Solution 2

Method logic: $$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0$$

- If s is replaced by $1/d$

$$\left(\frac{1}{d}\right)^n + a_{n-1}\left(\frac{1}{d}\right)^{n-1} + \cdots + a_1\left(\frac{1}{d}\right) + a_0 = 0$$

$$\left(\frac{1}{d}\right)^n \left[1 + a_{n-1}\left(\frac{1}{d}\right)^{-1} + \cdots + a_1\left(\frac{1}{d}\right)^{(1-n)} + a_0\left(\frac{1}{d}\right)^{-n}\right]$$
$$= \left(\frac{1}{d}\right)^n \left[1 + a_{n-1}d + \cdots + a_1 d^{(n-1)} + a_0 d^n\right] = 0$$

Case 1: Solution 2

Example: $$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

$$D(s) = 3s^5 + 5s^4 + 6s^3 + 3s^2 + 2s + 1$$

- **Unstable with 2 roots at the RHP**

s^5	3	6	2
s^4	5	3	1
s^3	4.2	1.4	
s^2	1.33	1	
s^1	−1.75		
s^0	1		

Case 2

Example: $$T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$

s^5	1	6	8
s^4	1	6	8
s^3	0	0	0
s^2			
s^1			
s^0			

$$P(s) = s^4 + 6s^2 + 8 \;\Rightarrow\; \frac{dP(s)}{ds} = 4s^3 + 12s + 0$$

Continue

Example: $$T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$

s^5	1	6	8
s^4	~~7~~ 1	~~42~~ 6	~~56~~ 8
s^3	~~0~~ ~~4~~ 1	~~0~~ ~~12~~ 3	~~0~~ ~~0~~ 0
s^2	3	8	0
s^1	$\frac{1}{3}$	0	0
s^0	8	0	0

$$P(s) = s^4 + 6s^2 + 8 \;\Rightarrow\; \frac{dP(s)}{ds} = 4s^3 + 12s + 0$$

- **Stable**

Case 2

- The order of auxiliary polynomial shows the number of poles on the jw-axis.

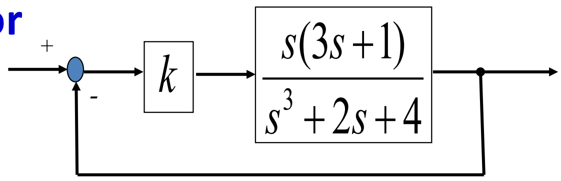
$$P(s) = s^4 + 6s^2 + 8$$

$$\Downarrow$$

4 poles on the jw-axis

Example

- Check the stability of following system for different values of k



$$M(s) = \frac{k\frac{s(3s+1)}{s^3+2s+4}}{1+k\frac{s(3s+1)}{s^3+2s+4}} = \frac{ks(3s+1)}{s^3+2s+4+ks(3s+1)} \Rightarrow s^3+3ks^2+(2+k)s+4=0$$

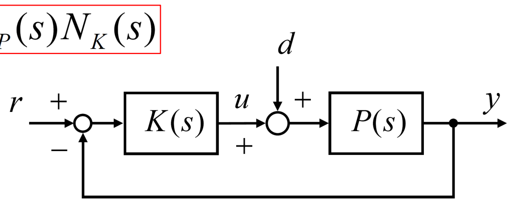
$$\begin{array}{lcc} s^3 & 1 & 2+k \\ s^2 & 3k & 4 \\ s^1 & \frac{3k(2+k)-4}{3k} & 0 \\ s^0 & 4 & 0 \end{array}$$

$$\Rightarrow \begin{array}{c} 3k>0 \\ \frac{3k^2+6k-4}{3k}>0 \end{array} \Rightarrow$$

For stability:
k>0.528

Pole-Zero Cancellation

$$P(s)=\frac{N_P(s)}{D_P(s)},\qquad K(s)=\frac{N_K(s)}{D_K(s)}\quad \Rightarrow \quad \phi(s):=D_P(s)D_K(s)+N_P(s)N_K(s)$$



$$G_{ur}=\frac{D_P(s)N_K(s)}{\phi(s)}\qquad G_{ud}=\frac{-N_P(s)N_K(s)}{\phi(s)}$$

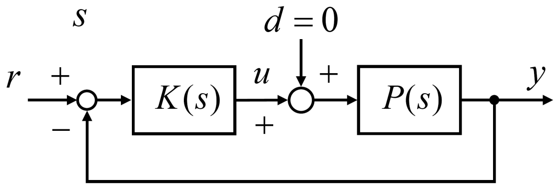
$$G_{yr}=\frac{N_P(s)N_K(s)}{\phi(s)}\qquad G_{yd}=\frac{N_P(s)D_K(s)}{\phi(s)}$$

Example: $P(s)=\dfrac{1}{s-1}\qquad K(s)=\dfrac{s-1}{s}$

$$\phi(s)=(s-1)\cdot s+1\cdot(s-1)\;=\underline{(s-1)}(s+1)=0\quad\Rightarrow\quad G_{yr}(s)=\frac{P(s)K(s)}{1+P(s)K(s)}=\frac{\cancel{s-1}}{(\cancel{s-1})(s+1)}$$

Continue

$$P(s)=\frac{1}{s-1},\;\; K(s)=\frac{s-1}{s}\;=1-\frac{1}{s}\quad\Rightarrow\quad P(s)K(s)=\frac{1}{\cancel{s-1}}\cdot\frac{\cancel{s-1}}{s}\;=\frac{1}{s}$$

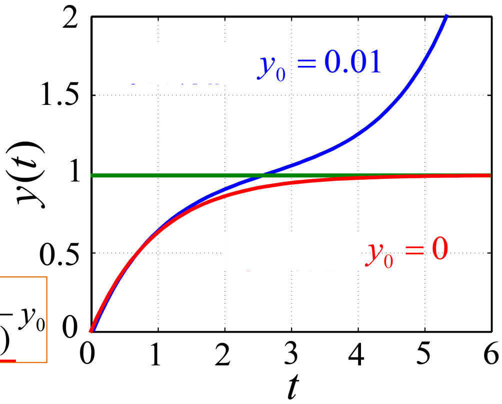


$$y(s)=\frac{P(s)K(s)}{1+P(s)K(s)}\cdot r(s)\;=\frac{\frac{1}{s}}{1+\frac{1}{s}}\cdot r(s)\;=\underline{\frac{1}{s+1}\cdot r(s)}$$

$$P(s)=\frac{y(s)}{u(s)}=\frac{1}{s-1}\quad\Rightarrow\quad \begin{array}{l}(s-1)y(s)=u(s)\\ \dot y(t)-y(t)=u(t)\end{array}$$

$$(sy(s)-y_0)-y(s)=u(s)\quad\Rightarrow\quad y(s)=\frac{1}{s-1}u(s)+\frac{1}{s-1}y_0$$

$$y(s)=\frac{1}{\cancel{s-1}}\cdot\frac{\cancel{s-1}}{s}(r(s)-y(s))+\frac{1}{s-1}y_0\Rightarrow y(s)=\frac{1}{s+1}r(s)+\frac{s}{(s+1)\underline{(s-1)}}y_0$$

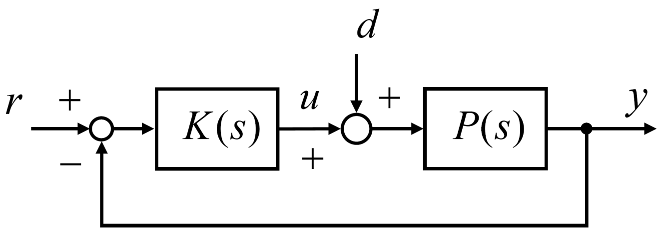


Internal Stability

Internal Stability: A system is internally stable if and only if the poles of all possible transfer functions have negative real parts.

$$G_{ur}(s)=\frac{K(s)}{1+P(s)K(s)} \qquad G_{ud}(s)=\frac{-P(s)K(s)}{1+P(s)K(s)}$$

$$G_{yr}(s)=\frac{P(s)K(s)}{1+P(s)K(s)} \qquad G_{yd}(s)=\frac{P(s)}{1+P(s)K(s)}$$



Asymptotic Stability

Asymptotic Stability: A system is asymptotically stable if and only if the all eigenvalues have negative real parts.

- How can we check asymptotic stability?

$$\text{Let } |sI-A|=0 \Rightarrow \text{Find eigenvalues } \lambda_1,\lambda_2,\ldots,\lambda_n$$

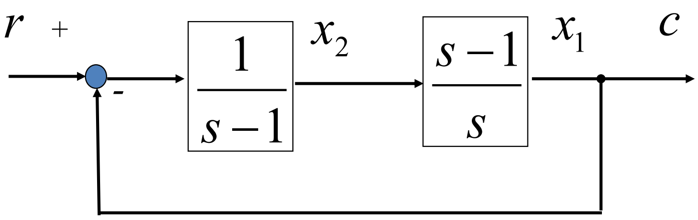
$$\text{If } \lambda_1,\lambda_2,\ldots,\lambda_n \in \text{LHP}$$

$$\Downarrow$$

System is asymptotically stable

Example

- **Check the Asymptotic/Internal stability of the following system:**



$$\dot{x}_2 - x_2 = r - x_1 \qquad \dot{x}_1 = \dot{x}_2 - x_2 =$$

$$\Rightarrow \quad \dot{x}_1 = -x_1 + r \qquad \dot{x}_2 = -x_1 + x_2 + r$$

$$\Rightarrow \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} r$$

$$|sI - A| = \begin{vmatrix} s+1 & 0 \\ 1 & s-1 \end{vmatrix} = s^2 - 1 = 0 \quad \Rightarrow \quad \lambda_1 = 1 \quad \lambda_2 = -1$$

$\Rightarrow$ **Not Asympt. Stability**

Thank You!