Linear Control System

Sensitivity Analysis

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Contents

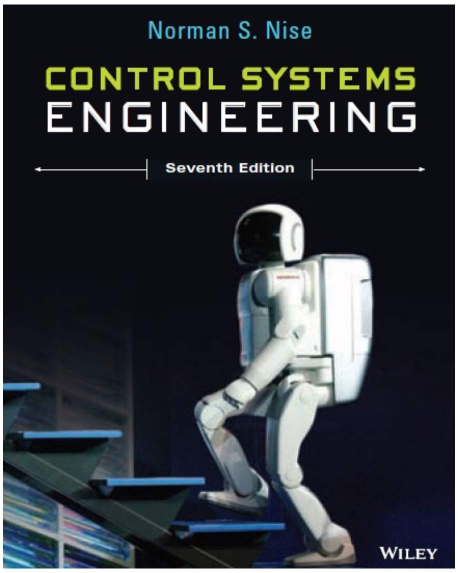
1. Sensitivity
2. Examples
3. Pole Sensitivity

Reference

1. Norman S. Nise, **Control Systems Engineering**, 7th Ed., Wiley, 2015.

Chapter 7, Section 7.7, pp.: 356-359

Chapter 8, Section 8.10, pp.: 415-422



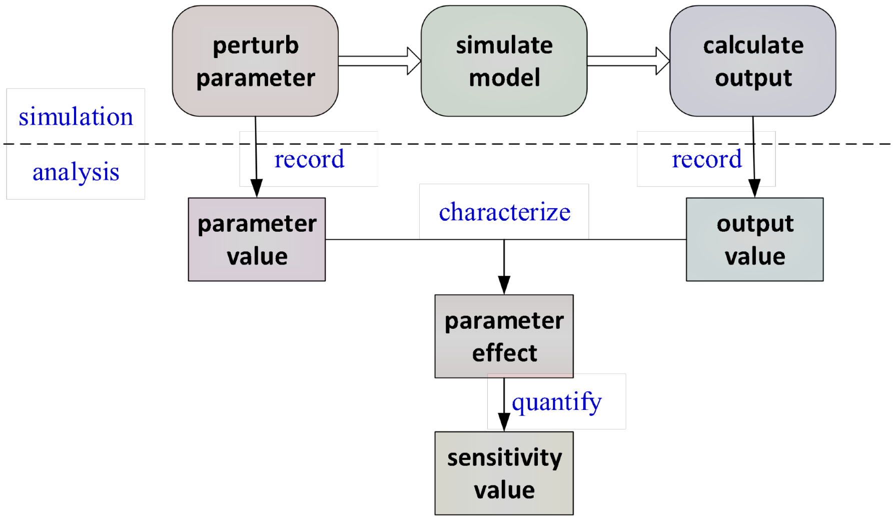
Sensitivity Analysis

- **Sensitivity analysis** aims to investigate how a model component affects another component/function

- **Sensitivity analysis** is an essential issue in mathematical modeling and analysis

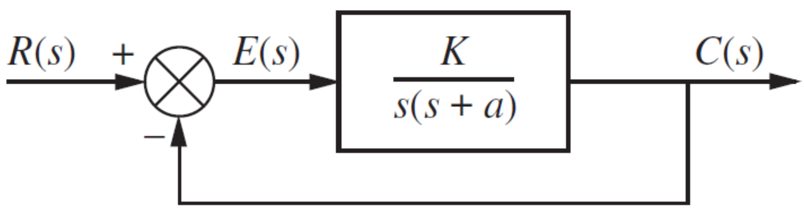
Sensitivity Analysis

- Sensitivity analysis can be considered as a systematic "**perturbation analysis**"



Sensitivity

- The degree to which changes in system parameters affect system transfer functions, and hence performance, is called **sensitivity**.



$S_{G:K}$ $S_{G:a}$ $S_{T:K}$ $S_{T:a}$ $S_{E:a}$ $S_{s:K}$

Sensitivity Formulation

- Sensitivity is the ratio of the fractional change in the function to the fractional change in the parameter as the fractional change of the parameter approaches zero.

$$S_{F:P} = \lim_{\Delta P \to 0} \frac{\text{Fractional change in the function, } F}{\text{Fractional change in the parameter, } P}$$

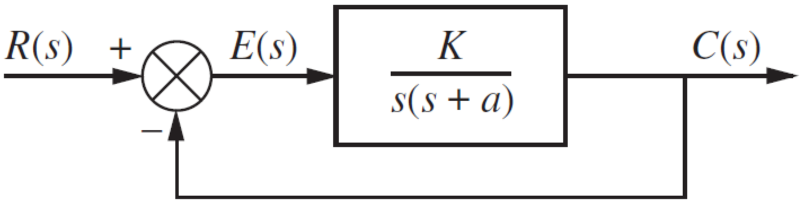
$$= \lim_{\Delta P \to 0} \frac{\Delta F / F}{\Delta P / P}$$

$$= \lim_{\Delta P \to 0} \frac{P \Delta F}{F \Delta P}$$

$$S_{F:P} = \frac{P}{F}\frac{\delta F}{\delta P}$$

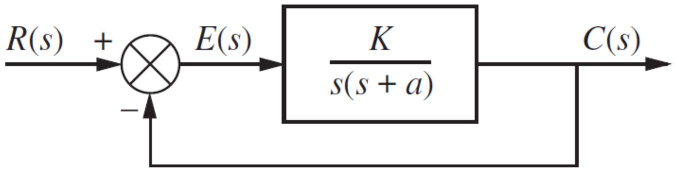
Example 1

- Find the sensitivity of the closed-loop transfer function to the parameter *a*



Solution

$$T(s) = \frac{K}{s^2 + as + K}$$

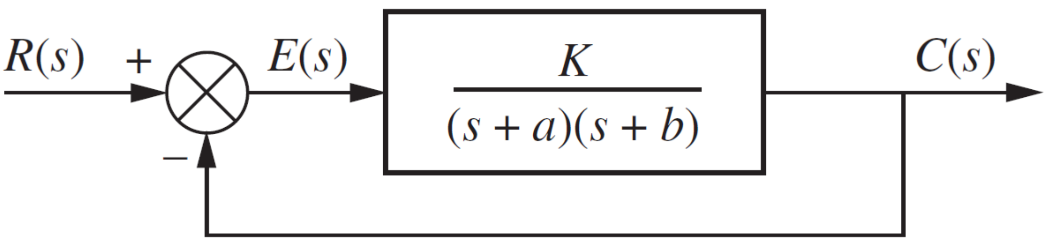


$$S_{T:a} = \frac{a}{T}\frac{\delta T}{\delta a} = \frac{a}{\left(\frac{K}{s^2 + as + K}\right)}\left(\frac{-Ks}{(s^2 + as + K)^2}\right) = \boxed{\frac{-as}{s^2 + as + K}}$$

An increase in ***K*** reduces the sensitivity of the closed-loop transfer function to changes in the parameter ***a***.

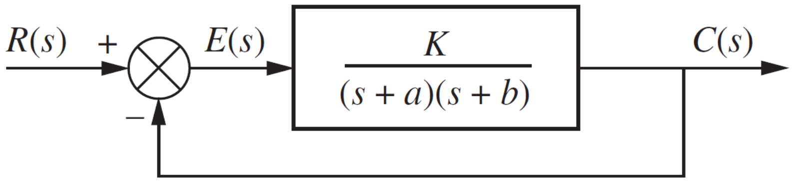
Example 2

- Find the sensitivity of the steady-state error to changes in parameter K and parameter ***a*** for the system with a step input.



Solution

- **The steady-state error:**



$$e(\infty) = \frac{1}{1+K_p} = \frac{1}{1+\frac{K}{ab}} = \frac{ab}{ab+K}$$

$$S_{e:a} = \frac{a}{e}\frac{\delta e}{\delta a} = \frac{a}{\left(\frac{ab}{ab+K}\right)} \frac{(ab+K)b - ab^2}{(ab+K)^2} = \boxed{\frac{K}{ab+K}}$$

$$S_{e:K} = \frac{K}{e}\frac{\delta e}{\delta K} = \frac{K}{\left(\frac{ab}{ab+K}\right)} \frac{-ab}{(ab+K)^2} = \boxed{\frac{-K}{ab+K}}$$

Pole Sensitivity

- The sensitivity of a closed-loop pole, ***s***, to gain, ***K***:

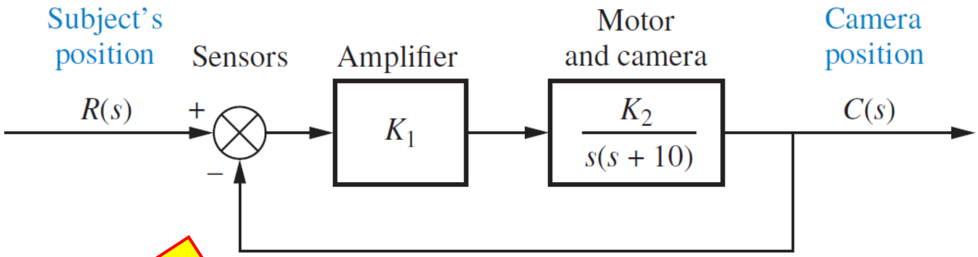
$$S_{s:K} = \frac{K}{s}\frac{\delta s}{\delta K}$$

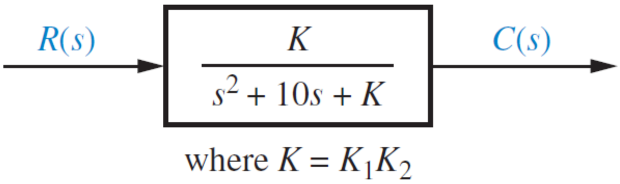
$$\Delta s = s(S_{s:K})\frac{\Delta K}{K}$$

Δs is the change in pole location, and $\Delta K/K$ is the fractional change in the gain, K.

Example 3: Security Camera

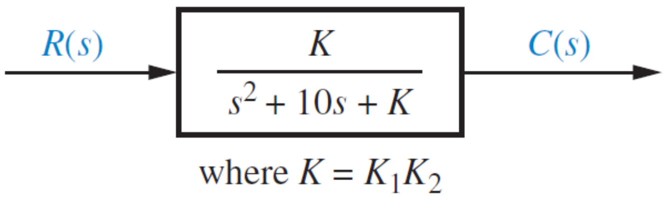
- Find the root sensitivity of the system at $s = -9.47$ and $s=-5+j5$. Also calculate the change in the pole location for a 10% change in K.





Solution

Differentiating the Characteristic equation with respect to K:



$$2s\frac{\delta s}{\delta K} + 10\frac{\delta s}{\delta K} + 1 = 0 \quad \Rightarrow \quad \frac{\delta s}{\delta K} = \frac{-1}{2s+10}$$

$$S_{s:K} = \frac{K}{s}\frac{\delta s}{\delta K} \quad \Rightarrow \quad S_{s:K} = \frac{K}{s} * \frac{-1}{2s+10}$$

Continue

$$S_{s:K} = \frac{K}{s} * \frac{-1}{2s+10}$$

For ***s* = -9.47**, from Table: ***K* = 5.**

$$S_{s:K} = -0.059$$

K	Pole 1	Pole 2
0	−10	0
5	−9.47	−0.53
10	−8.87	−1.13
15	−8.16	−1.84
20	−7.24	−2.76
25	−5	−5
30	$-5+j2.24$	$-5-j2.24$
35	$-5+j3.16$	$-5-j3.16$
40	$-5+j3.87$	$-5-j3.87$
45	$-5+j4.47$	$-5-j4.47$
50	$-5+j5$	$-5-j5$

The change in the pole location for a 10% change in *K*, with $s = -9.47$; $\Delta K/K = 0.1$, and $S_{s:K} = -0.059$:

$$\Delta s = s(S_{s:K})\frac{\Delta K}{K}$$

$$\Delta s = 0.056$$

The pole will move to the right by **0.056** units for a 10% change in *K*.

Continue

For $s = -5 + j5,\ K = 50.$

$$S_{s:K} = \frac{K}{s} * \frac{-1}{2s+10}$$

$$S_{s:K} = 1/(1+j1) = (1/\sqrt{2})\angle -45^\circ$$

K	Pole 1	Pole 2
0	−10	0
5	−9.47	−0.53
10	−8.87	−1.13
15	−8.16	−1.84
20	−7.24	−2.76
25	−5	−5
30	$-5+j2.24$	$-5-j2.24$
35	$-5+j3.16$	$-5-j3.16$
40	$-5+j3.87$	$-5-j3.87$
45	$-5+j4.47$	$-5-j4.47$
50	$-5+j5$	$-5-j5$

The change in the pole location for a 10% change in *K*, with:

$$s = -5 + j5,\ \Delta K/K = 0.1,\ S_{s:K} = (1/\sqrt{2})\angle -45^\circ$$

$$\Delta s = s(S_{s:K})\frac{\Delta K}{K}$$

$$\Delta s = -j5$$

The pole will move vertically by **5** units for a 10% change in *K*.

Thank You!